KARADENİZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

MATEMATİK ANABİLİM DALI

# REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU

YÜKSEK LİSANS TEZİ

Çiğdem GEBİÇ

HAZİRAN 2022
TRABZON

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**Çiğdem GEBİÇ**

**Karadeniz Teknik Üniversitesi Fen Bilimleri Enstitüsünce**
**"YÜKSEK LİSANS (MATEMATİK)"**
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# ÖNSÖZ

Tez konumun belirlenmesinden, çalışmamın tamamlanıp bu hale getirilmesine kadar geçen süreçte benden desteğini ve yardımını esirgemeyen, lisans ve yüksek lisans öğrenimim boyunca kendisinden, araştırmalarından, çalışmalarından ve kişiliğinden çok şey öğrendiğim sayın Prof. Dr. Haskız COŞKUN hocama emeği için teşekkür eder, saygılarımı sunarım.

Tez çalışmamı hazırlarken bana yardım eden değerli dostlarıma, hayatım boyunca yanımda olan ve her türlü kararımda maddi ve manevi destek sağlayan, bu günlere gelmemde emeği olan anne ve babama, beni her konuda cesaretlendiren, motive eden kardeşlerime teşekkürü bir borç bilirim.

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**TEZ ETİK BEYANNAMESİ**

Yüksek Lisans Tezi olarak sunduğum “REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU” başlıklı bu çalışmayı baştan sona kadar danışmanım Prof.Dr. Haskız COŞKUN’un sorumluluğunda tamamladığımı, verileri/örnekleri kendim topladığımı, deneyleri/analizleri ilgili laboratuvarlarda yaptığımı/yaptırdığımı, başka kaynakladan aldığım bilgileri metinde ve kaynakçada eksiksiz olarak gösterdiğimi, çalışma sürecinde bilimsel araştırma ve etik kurallarına uygun olarak davrandığımı ve aksinin ortaya çıkması durumunda her türlü yasal sonucu kabul ettiğimi beyan ederim. 14/06/2022

Çiğdem GEBİÇ

# İÇİNDEKİLER

**Sayfa No**

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ÖZET

REGÜLER SINIR DEĞER PROBLEMİNİN GREEN FONKSİYONU

Çiğdem GEBİÇ

Karadeniz Teknik Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı
Danışman: Prof. Dr. Haskız COŞKUN
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Bu çalışma esas olarak üç bölümden oluşmaktadır.

Birinci bölümde, temel bazı bilgiler yardımıyla sınır değer problemleri için Green fonksiyonlarının elde edilişi incelenmiştir.

İkinci bölümde, $q(x) \in L^1[0,\pi]$ olmak üzere

$$y'' + (\lambda - q)y = 0 \quad ,0 \le x \le \pi$$

$$\frac{y'(0,\lambda)}{y(0,\lambda)} = -tan\alpha,$$

$$cos\beta y'(\pi,\lambda) + sin\beta y(\pi,\lambda) = 0$$

probleminin Green fonksiyonları asimptotik olarak bulunmuştur. Dirichlet ve Neumann sınır koşullarının sağlanması durumunda da Green fonksiyonu hesaplanmıştır.

Üçüncü bölümde ise potansiyel fonksiyonu

$q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$ olması durumu ele alınarak bazı özel başlangıç değerlerini sağlayan çözümler için asimptotik tahminler elde edilmiş ve bu tahminler kullanılarak Green fonksiyonları hesaplanmıştır. Ayrıca Dirichlet ve Neumann sınır koşulları durumunda Green fonksiyonları asimptotik olarak elde edilmiştir.

**Anahtar Kelimeler:** Green fonksiyonu, Asimptotik çözümler, Dirichlet ve Neumann sınır koşulları

Master Thesis

SUMMARY

THE GREEN'S FUNCTION OF REGULAR BOUNDARY VALUE PROBLEM

Çiğdem GEBİÇ

Karadeniz Technical University
The Graduate School of Natural and Applied Sciences
Mathematics Graduate Program
Supervisor: Professor. Haskız COŞKUN
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This study basically consists of three sections. In the first section, we formulate Green's functions for boundary value problems based on some main theory. In the second section, e consider the boundary value problem

$$y'' + (\lambda - q)y = 0 \quad ,0 \leq x \leq \pi$$

$$\frac{y'(0,\lambda)}{y(0,\lambda)} = -tan\alpha,$$

$$cos\beta y'(\pi,\lambda) + sin\beta y(\pi,\lambda) = 0$$

and compute Green's functions associuted with it. In addition to these, we also consider Dirichlet and Neumann boundary conditions.

In the last section, we particularly consider the special integrable potential

$q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$, and find asymptotic estimates for the solutions satisfying certain initial values, and then using these estimates we evaluate Green's functions.

**Key Words:** Green's functions, Asymptotic solutions, Dirichlet and Neumann boundary conditions

## SEMBOLLER DİZİNİ

| | |
|---|---|
| $\lambda$ | : özdeğer |
| $\lambda \to \infty$ | : $\lambda$, $\infty$'a yaklaşırken |
| $W(u,v)(x)$ | : Wronskian determinantı |
| $G(x,y,\lambda)$ | : Green fonksiyonu |
| π | : Pi sayısı, yaklaşık değeri 3,14159265 |
| O,o | : asimptotik terimleri gösteren simgeler |
| $W(u,v)(x)$ | : Wronskian determinantı |
| $G(x,y,\lambda)$ | : Green fonksiyonu |

# 1.GENEL BİLGİLER

## 1.1.Giriş

İkinci mertebeden lineer bir diferansiyel denklem genel olarak aşağıdaki şekilde ifade edilir.

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x), \quad x_1 < x < x_2 \quad (A_2(x) \neq 0) \tag{1.1}$$

Birçok uygulamada, (1.1) denkleminin bazı yardımcı koşulları sağlaması istenir. Bunların sayısı denklemin mertebesi kadardır. Eğer verilen koşullar tek noktada tanımlı ise bu koşullara başlangıç koşulları, iki noktada tanımlı ise bu koşullara sınır koşulları denir. Sınır koşulları genel olarak

$$\left.\begin{aligned} a_{11}y(x_1) + a_{12}y'(x_1) + b_{11}y(x_2) + b_{12}y'(x_2) = \alpha \\ a_{21}y(x_2) + a_{22}y'(x_2) + b_{21}y(x_2) + b_{22}y'(x_2) = \beta \end{aligned}\right\} \tag{1.2}$$

şeklinde ifade edilir. Burada $a_{ij}, b_{ij}, \alpha, \beta$ sabitlerdir. (1.1) denkleminin (1.2) koşullarını sağlayan çözümünü bulma problemine sınır değer problemi denir. Uygulamada kullanılan bazı sınır koşulları (1.2)'nin bir özel durumu olan ayrılmış sınır koşullarıdır. Yani birinci denklem sadece $x_1$'e ve ikinci denklem $x_2$'ye bağlıdır. Yani

$$\left.\begin{aligned} a_{11}y(x_1) + a_{12}y'(x_1) = \alpha, \qquad a_{11}^2 + a_{12}^2 \neq 0 \\ a_{21}y(x_2) + a_{22}y'(x_2) = \beta, \qquad a_{21}^2 + a_{22}^2 \neq 0 \end{aligned}\right\} \tag{1.3}$$

(1.2) koşulları bazen ayrılmamış sınır koşulları olarak da adlandırılır. Bunlar uygulamada ayrılmış sınır koşulları kadar yaygın değildir.

Ayrıca F(x) = 0 ise (1.1) denklemine homojendir denir. Eğer $\alpha = \beta = 0$ ise sınır koşullarına da homojen sınır koşulları adı verilir.

Sınır değer problemlerinin özel bir sınıfa ait olan özdeğer problemleri uygulamada özellikle inşaat, uçak, gemi tasarımlarında da kullanılmaktadır.

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x), \quad x_1 < x < x_2 \tag{1.4}$$

$$\left.\begin{array}{ll} {}_{11}y(x_1) + \ {}_{12}y'(x_1) = \alpha, & a_{11}^2 + \ {}_{12}^{2} \neq 0 \\ {}_{21}y(x_2) + \ {}_{22}y'(x_2) = \beta, & a_{21}^2 + \ {}_{22}^{2} \neq 0 \end{array}\right\} \tag{1.5}$$

problemini ele alalım.

$$M \coloneqq A_2(x)D^2 + A_1(x)D + A_0(x) \tag{1.6}$$

olarak tanımlanırsa (1.4) denklemi

$$M[y] = F(x)$$

olarak yazılabilir. Benzer şekilde sınır operatörleri $B_1$ ve $B_2$ aşağıdaki şekilde tanımlanırsa

$$B_1[y] = \ {}_{11}y(x_1) + \ {}_{12}y'(x_1) \tag{1.7}$$

$$B_2[y] = \ {}_{21}y(x_2) + \ {}_{22}y'(x_2) \tag{1.8}$$

bu durumda (1.5) eşitlikleri

$$B_1[y] = \alpha, \;\; B_2[y] = \beta \tag{1.9}$$

olarak kısaca kompakt bir formda ifade edilmiş olur.

$$M[c_1 f + c_2 g] = c_1 M[f] + c_2 M[g]$$

$(c_1, c_2$ sabitler) sağlanırsa operatöre lineerdir denir.

Daha genel olarak

$$L[y] + \lambda r(x) y = 0, \; x_1 < x < x_2$$

$$\left.\begin{array}{ll} B_1[y] = \ {}_{11}y(x_1) + \ {}_{12}y'(x_1) = \alpha, & a_{11}^2 + \ {}_{12}^{2} \neq 0 \\ B_2[y] = \ {}_{21}y(x_2) + \ {}_{22}y'(x_2) = \beta, & a_{21}^2 + \ {}_{22}^{2} \neq 0 \end{array}\right\}$$

$$L = D[P(x)D] + q(x)$$

sınıfına ait problemler özdeğer problemleri olarak adlandırılır. Özel olarak

$$y(x_1) = y(x_2) = 0$$

durumu Dirichlet sınır koşulları,

$$y'(x_1) = y'(x_2) = 0$$

Neumann sınır koşulları olarak bilinmektedir.

Bilindiği üzere homojen olmayan başlangıç değer ve sınır değer problemleri için Green fonksiyonları yardımıyla özel çözümler bulunabilir. Sınır değer problemleri için Green fonksiyonları başlangıç değer problemlerindekine benzer olarak tanımlanmasına rağmen sınır değer problemleri için Green fonksiyonlarının her zaman mevcut olmadığı bilinmektedir. Oysa başlangıç değer problemleri için Green fonksiyonları daima bulunabilir.

### 1.2. Sınır Değer Problemleri İçin Green Fonksiyonları

Öncelikle

$$M[y] = F(x), \qquad x_1 < x < x_2 \tag{1.10}$$

$$B_1[y] = a_{11}y(x_1) + a_{12}y'(x_1) = \alpha$$

$$B_2[y] = a_{21}y(x_2) + a_{22}y'(x_2) = \beta$$

sınır değer problemini ele alalım. Burada M operatörü

$M = A_2(x)D^2 + A_1(x)D + A_0(x),\ D = \frac{d}{dx}$ şeklindedir. (1.10) problemi için Green fonksiyonlarını elde etmeden önce (1.10) problemi kendine eş forma konulur. Hatırlanacağı üzere $M[y] = F(x)$ kendine eş formda değilse $\left(A_2'(x) \neq A_1(x)\right)$

$$p(x) = e^{\int A_1(x)/A_2(x)dx}$$

olmak üzere,

$$\mu(x) = p(x)/A_2(x)$$

ile denklemin her iki tarafı çarpılarak denklem kendine eş forma konur. Bu durumda (1.10) denkleminin operatörü L ile gösterilir ve L kendine eştir denilir. Bu durumda ilgili problem

$$L[y] = f(x), \quad x_1 < x < x_2 \quad , \qquad B_1[y] = \alpha\,, \qquad B_2[y] = \beta \tag{1.11}$$

$$f(x) = {p(x)F(x)}\big/{A_2(x)}\,,$$

$$q(x) = {p(x)A_0(x)}\big/{A_2(x)}$$

şeklinde düzenlenirse

$$L = D[P(x)D] + q(x)$$

olarak ifade edilir. Hatırlanacağı üzere (1.11) denkleminin genel çözümü

$$y = y_H + y_p$$

şeklindedir. Burada $y_H$

$$L[y] = 0 \quad , \quad B_1[y] = \alpha, \;\; B_2[y] = \beta \tag{1.12}$$

ve $y_p$ ise

$$L[y] = f(x) \qquad , \qquad B_1[y] = 0, \qquad B_2[y] = 0 \tag{1.13}$$

denklemin bir özel çözümüdür. (1.12)' nin genel çözümü

$$y_H = c_1 y_1(x) + c_2 y_2(x)$$

şeklindedir. Burada $y_1 \; ve \; y_2$ homojen denklemin lineer bağımsız çözümleri, $c_1 \; ve \; c_2$ homojen olmayan sınır koşullarını sağlayacak şekilde belirlenen sabitlerdir. Şimdi (1.13) problemini ele alalım. Başlangıç değer problemlerindekine benzer olarak

$$y_p = -\int_{x_1}^{x_2} g(x,s) f(s) ds \tag{1.14}$$

şeklinde bir çözümün varlığı kabul edilir. Burada $g(x,s)$ belirlenecek Green fonksiyonunu göstermektedir. ( (-) işareti ise $g(x,s)$ nin özel bir fiziksel yoruma sahip olması içindir.) Hesaplamalarda Dirac delta fonksiyonundan yararlanacağız.

### 1.3. Dirac Delta Fonksiyonu

$g(t)$: $(-\infty, +\infty)$ üzerinde sürekli olmak üzere

$$t \neq a \text{ için } \delta(t-a) = 0,$$

$$\int_{-\infty}^{+\infty} \delta(t-a)dt = 1,$$

$$\int_{-\infty}^{+\infty} g(t)\delta(t-a)dt = g(a),$$

$$\int_{-\infty}^{+\infty} \delta(u-a)du = \begin{cases} 1, t > a \\ 0, t < a \end{cases} = H(t-a)$$

özelliklerini sağlar.

(1.14)'ün her iki tarafına L operatörü uygulanırsa ve L operatörünün integral ile yer değiştirebileceği varsayılırsa

$$L[y_p] = L\left[-\int_{x_1}^{x_2} g(x,s)f(s)\,ds\right] = -\int_{x_1}^{x_2} L[g]f(s)ds = f(x)$$

dir. Son eşitliğin sağ tarafının $f(x)$'e eşit olabilmesi için

$$L[g] = -\delta(x-s) \qquad , \qquad x_1 < x < x_2 \tag{1.15}$$

sağlanmalıdır. ( Burada $\delta(x-s)$ dirac delta fonksiyonunun

$\int_{-\infty}^{+\infty} \delta(t-a)f(t)dt = f(a)$ özelliği kullanılmıştır.) $g(x,s)$' yi belirlemek için ise (1.15)'den başka bazı koşullar gereklidir. Bunun için (1.13)'deki sınır koşulları (1.14)'e uygulanırsa

$$B_1[y_p] = -\int_{x_1}^{x_2} B_1[g]f(s)ds = 0,$$

$$B_2[y_p] = -\int_{x_1}^{x_2} B_2[g]f(s)ds = 0$$

bulunur. $f$ fonksiyonu herhangi bir fonksiyon olabileceğinden bu son eşitliklerin sağlanabilmesi için

$$B_1[g] = 0, \qquad B_2[g] = 0 \tag{1.16}$$

sağlanmalıdır. Sonuç olarak aradığımız Green fonksiyonu

$$L[g] = -\delta(x-s), \qquad x_1 < x < x_2, \qquad B_1[g] = 0, \qquad B_2[g] = 0 \tag{1.17}$$

probleminin çözümüdür. Bu problem (1.13)'e benzerdir. Burada $x_1 < s < x_2$ ve s sabittir. (1.17) problemi (1.13) problemine benzemesine rağmen (1.17)'deki kuvvet fonksiyonu dirac delta fonksiyonudur. Bunun anlamı ise (1.17)'deki g yi bulma problemi (1.13)'deki (y)'yi bulma probleminden daha basittir. Böylece L operatörü ve özel sınır koşullarıyla ilgili Green fonksiyonu bulunduktan sonra bu Green fonksiyonu (1.13)'ü çözmek için kullanılır.

(1.17)'deki delta fonksiyonu, $g(x,s)$'nin $x = s$ civarında detaylı bir biçimde incelenmesi gerekliliğini ortaya koyar. Bunun için de (1.14) göz önüne alınırsa

$$y_p(s^+) - y_p(s^-) = -\int_{x_1}^{x_2} [g(s^+,s) - g(s^-,s)]f(s)ds$$

( Burada $y(s^+) = \lim_{\varepsilon\to 0^+} y(s+\varepsilon)$, $\quad y(s^-) = \lim_{\varepsilon\to 0^-} y(s-\varepsilon)$ )

bulunur. Fakat bir diferansiyel denklemin çözümü sürekli bir fonksiyon olacağı için son eşitliğin sol tarafı sıfır olur ve f keyfi olduğundan

$$g(s^+,s) = g(s^-,s) \tag{1.18}$$

bulunur. Bu ise g nin $x = s$ de sürekli olması gerektiğini belirtir. Bir adım daha öteye gidilirse, $g(x,s)$ nin türevinin incelenmesi gerekir. Bunun için $L[g] = -\delta(x-s)$, $[s^-,s^+]$ da x'e göre integre edilirse

$$\left(L = D[P(x)D] + q, \qquad L[g] = \int_{s^-}^{s^+} D[P(x)D](g) + \int_{s^-}^{s^+} qg\right) = -\int\limits_{s^-}^{s^+} \delta(x-s)dx$$

$$\left[P(x)\frac{\partial g}{\partial x}\right]\Big|_{x\,=\,s^-}^{\;s^+} + \int_{s^-}^{s^+} q(x)g(x,s)dx \;= \int_{s^-}^{s^+} \delta(x-s)dx.$$

q(x) ve $g(x,s)$ nin $x=s$'de sürekliliğinden sol taraftaki integralin

$$\int_{s^-}^{s^+} q(x)g(x,s)dx = 0$$

olduğu gözlenir. Ayrıca $\delta$ fonksiyonları için integral özellikleri kullanılarak ve p(x)'in sürekli ve $[x_1, x_2]$ üzerinde sıfırdan farklılığı da dikkate alınırsa son eşitlik

$$\left(-\int_{s^-}^{s^+} \delta(x-s)dx\right) = 1,$$

$$\left.\frac{\partial g}{\partial x}\right|_{x\,=\,s^-}^{s^+} = -\frac{1}{p(s)}\,. \tag{1.19}$$

Bu sonuç ise $x=s$ de $g(x,s)$ nin büyüklüğü $\frac{1}{p(s)}$ olan bir sıçrama süreksizliğine sahip olduğunu gösterir. Böylece Green fonksiyonu şöyle tanımlanır.

**Tanım 1.1:**

$$L[y] = f(x), \quad x_1 < x < x_2, \qquad B_1[y] = \alpha, \quad B_2[y] = \beta$$

sınır değer problemi için $g(x,s)$ Green fonksiyonu aşağıdaki özellikleri sahiptir.

$(L = D[P(x)D] + q(x))\;\; (x_1 < x < x_2)$

$a)\; L[g] = -\delta(x-s),\; x_1 < x < x_2\;\; (s\; sabit)$

$b)\; B_1[g] = 0,\; B_2[g] = 0$

$c)\; g(s^+, s) = g(s^-, s)$

$d)\; \left.\frac{\partial g}{\partial x}\right|_{x\,=\,s^-}^{s^+} = -\frac{1}{p(s)}.$

Bu tanıma bağlı olarak Green fonksiyonu aşağıdaki şekilde elde edilir. Öncelikle (a)'dan ya $x < s$ veya $x > s$ olduğu görülür. O zaman $\delta$ fonksiyonu tanımından $L[g] = 0$ elde edilir. Çünkü her ikisinde $x \neq s$ dir. Sonra, eğer $z_1\; ve\; z_2$ $L[g] = 0$ homojen denkleminin $B_1[z_1] = 0, B_2[z_2] = 0'\imath$ sağlarsa ve çözümleri ise bu durumda a) ve b) den

$$g(x,s) = \begin{cases} u(s)z_1(x), & x < s \\ v(s)z_2(x), & x > s \end{cases} \tag{1.20}$$

bulunur. Burada $u, v$ belirlenecek fonksiyonlardır. Tanımdaki (c) ve (d) kullanılırsa $u$ ve $v$ fonksiyonları

$$\begin{aligned} v(s)z_2(s) - u(s)z_1(s) &= 0 \\ v(s)z_2'(s) - u(s)z_1'(s) &= -\frac{1}{p(s)} \end{aligned} \tag{1.21}$$

ve (1.21)'i aynı anda sağlayan $u(s)$ ve $v(s)$

$$\begin{aligned} u(s) &= -\frac{z_2(s)}{p(s)W(z_1,z_2)(s)} \\ v(s) &= -\frac{z_1(s)}{p(s)W(z_1,z_2)(s)} \end{aligned} \tag{1.22}$$

bulunur. Burada

$$W(z_1,z_2) = z_1 z_2' - z_1' z_2$$

Wronskian fonksiyonudur.

Bu arada şu önemli noktayı vurgulamak istiyoruz. Eğer $y_1$ ve $y_2$ aynı diferansiyel denklemin çözümleri iseler,

$$P(x)W(z_1,z_2)(s) = p(x)W(z_1,z_2)(x) = c$$

ve bu durumda (1.21) deki Green fonksiyonu

$$g(x,s) = \begin{cases} -z_1(x)z_2(s)/c\,, & x_1 < x < s \\ -\,z_1(s)z_2(x)/c\,, & s \le x < x_2 \end{cases} \quad (x\ değişken\ ve\ s\ sabit)$$

veya denk olarak

$$g(x,s) = \begin{cases} -z_1(s)z_2(x)/c\,, & x_1 < s \le x \\ -z_1(x)z_2(s)/c\,, & x_1 < s < x_2 \end{cases} \qquad (1.23)\ \ (x\ sabit\ ve\ s\ değişken)$$

elde edilir. $c$ bir sabit olduğundan (1.23)'den Green fonksiyonunun $x$ ve $s$'ye göre simetrik olduğu gözlenir. Yani $g(x,s) = g(s,x)$'dir. Bu özellik, Green fonksiyonunun diğer yöntemlerle elde edişinde önemli rol oynar.

### 1.4. Çözümlü Alıştırmalar

**Örnek 1.1:**

$$y'' + k^2 y = f(x), \quad 0 < x < 1, \quad y(0) = 0, \qquad y(1) = 0, \ k \neq 0$$

sınır değer problemi için Green fonksiyonunu elde ediniz.

**Çözüm:**

İlgili homojen denklemin lineer bağımsız çözümleri $y_1 = coskx, \ y_2 = sinkx$. $z_1$ ve $z_2$ çözümünü bulmak için önce $z_1$ ve $z_2$ nin lineer kombinasyonu olduğu kabul edilir.

$$z_1 = c_{11} coskx + c_{12} sinkx$$

şeklinde kabul edilir. $z_1(0) = 0$ dan $c_{11} = 0$ ve $c_{12} = 1$ alınırsa

$$z_1 = sinkx,$$

benzer şekilde

$z_2 = c_{21} coskx + c_{22} sinkx$ ve $z_2(1) = 0$'dan

$$c_{21} cosk + c_{22} sink = 0.$$

$c_{11}$ ve $c_{22}$'yi öyle seçmeliyiz ki son eşitlik sağlanmalı ve aynı anda $c_{21}$ ve $c_{22}$ ikisi birden sıfır olmamalıdır. Diğer seçimler arasında $c_{21} = sink$ ve $c_{22} = -cosk$,

$$z_2 = sinkcoskx - cosksinkx = sink(1-x)$$

bulunur. Son olarak c sabitini belirlemeliyiz.

Öncelikle $L = D^2 + k^2$ kendine eş bir formda ve $P(x) = 1$ olduğunu gözleriz. Buradan

$$c = P(x)W(z_1, z_2)(x) = \begin{vmatrix} sinkx & sink(1-x) \\ kcoskx & -kcosk(1-x) \end{vmatrix}$$

ve $c = -ksink$

bulunur. Green fonksiyonu c ile bölüneceğinden $k \neq n\pi \ (n = 1,2,3, \dots)$ olmalıdır. Bu durumda

$$g(x,s)=\begin{cases}\dfrac{sinks\times sink(1-x)}{ksink}, & 0<s\leq x\\ \dfrac{sinkx\times sink(1-s)}{ksink}, & x<s<1\end{cases}$$

Eğer $k=n\pi\ (n=1,2,3,\dots)$ izin verilirse $c=0$ bu durumda Green fonksiyonu elde edilemez. Bu durumda

$$z_1=z_2=sinn\pi x$$

ki bunun anlamı

$$y''+n^2\pi^2y=0,\ \ y(0)=0,\ \ y(1)=0$$

probleminin sıfırdan farklı çözümü olduğu anlaşılır.

Bu örnekten anlaşılacağı üzere sınır değer problemi için Green fonksiyonu her zaman mevcut değildir. Yani $c=0$ ise $g(x,s)$ elde edilemez. Bu noktayı vurgulamak için aşağıdaki daha genel problemi ele alalım.

$$\left.\begin{array}{ll}L[y]=f(x) & x_1<x<x_2,\\ B_1[y]=\alpha, & B_2[y]=\beta\end{array}\right\} \qquad (1.24)$$

Bilindiği üzere (1.24)'ün çözümü için aşağıdaki iki problem oluşturulur:

***i***) $L[y]=0,\quad B_1[y]=\alpha,\quad B_2[y]=\beta,$

***ii***) $L[y]=f(x),\quad B_1[y]=0,\quad B_2[y]=0.$

(1.24)'ün çözümü (i) ve (ii) den elde edilen çözümlerin toplamıdır.

ii) problemini çözmek için Green fonksiyonu araştırılır. Ancak Green fonksiyonun mevcut olabilmesi için

$$L[y]=0,\quad B_1[y]=0,\quad B_2[y]=0$$

sınır değer problemi sadece trivial çözüme sahip olmalıdır. Sıfırdan farklı çözüme sahipse $z_1$ ve $z_2$ lineer bağımlıdır ve böylece $W(z_1,z_2)=0$ veya $c=0$ dır. Bu durumda Green fonksiyonları elde edilemez. Eğer (2) sadece trivial çözüme sahipse $z_1$ ve $z_2$ lineer bağımsız olacağından $c\neq 0$ ve Green fonksiyonu tek türlü bellidir.

**Örnek 1.2:**

$$y'' + y = sinx, \quad 0 < x < \frac{\pi}{2},$$

$$y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$$

problemini Green fonksiyonunu kullanarak çözünüz.

**Çözüm:**

$\boldsymbol{i}$) $\;y'' + y = 0 \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$

$\boldsymbol{ii}$) $\;y'' + y = sinx, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0$

Birinci problem $y_H = c_1 cosx + c_2 sinx$

$$y(0) = 0 \Rightarrow c_1 = 1$$

$$y\left(\frac{\pi}{2}\right) = -1 \Rightarrow c_2 = -1$$

$$y = cosx - sinx.$$

Homojen denklemin lineer bağımsız çözümleri ( homojen sınır koşullarını sağlayan)

$$z_1 = sinx, \quad z_2 = cosx.$$

alınabilir. $P(x) = 1$ olduğundan

$$P(x)W(z_1, z_2)(x) = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix} = -1$$

$$c = -1$$

$$g(x,s) = \begin{cases} sinscosx, & 0 < s \leq x \\ sinxcoss, & x < s < {}^{\pi}/_{2}. \end{cases}$$

Yukarıdaki ikinci problem kendine eş formdadır. Böylece $f(x) = sinx$ alınırsa

$$y_p = -\int_0^{\pi/2} g(x,s) sinsds = -cosx \int_0^x sin^2 sds - sinx \int_x^{\pi/2} cosssinsds$$

$$_p = -\frac{1}{2}xcosx$$

$$= \;_H + \;_p = \left(1 - \frac{x}{2}\right) cosx - sinx.$$

**Örnek 1.3:**

Green fonksiyon yöntemini kullanarak

$$2x^2 \;'' + 3x \;' - \; = 4x^{3/2}, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0$$

problemini çözünüz.

**Çözüm:**

$$_H = 2x^2 \;'' + 3x \;' - \; = 0, \quad y(0) = 0, \quad y(1) = 0$$

probleminin çözümüdür. Bu bir Cauchy Euler denklemidir. Bu denklemin genel çözümü

$$_H = c_1x^{1/2} + c_2x^{-1}.$$

$y(0) = 0$ sağlanması için $c_2 = 0$ olmalı ve $c_1$ keyfi olmalıdır.

İkinci sınır koşulundan ise $y(1) = c_1 = 0$'dır. Böylece

$$_H = 0.$$

Bu ise orijinal problemin bir tek çözümü olabilmesi için gereklidir. $y_p$'yi çözmek için önce denklem kendine eş forma konur, $P(x)$ ve $f(x)$ bulunur.

$$P(x) = e^{\int \frac{3x}{2x^2}dx} = x^{3/2}$$

Böylece diferansiyel denklem $2x^2$ ile bölünür ve $P(x) = x^{3/2}$ ile çarpılırsa kendine eş denklem

$$\frac{d}{dx}\left(x^{3/2} \;'\right) - \frac{1}{2}x^{-1/2} \; = 2x \quad y(0) = 0, \quad y(1) = 0.$$

elde edilir. Böylece $x^{1/2}$ ve $x^{-1}$ Green fonksiyonlarını oluşturmak için kullanılır. $x^{1/2}$ ve $x^{-1}$ in lineer kombinasyonları oluşturulursa

$$z_1 = c_{11}x^{1/2} + c_{12}x^{-1}$$

$$z_2 = c_{21}x^{1/2} + c_{22}x^{-1}$$

sınır koşulları kullanılırsa

$z_1(0) = 0$'dan $c_{12} = 0$, $c_{11}$ keyfidir. Diyelim ki $c_{11} = 1$ olsun ve böylece $z_1 = x^{1/2}$ olur.

$z_2(1) = 0$'dan $c_{21} + c_{22} = 0$. $c_{21} = -c_{22} = 1$ ve $z_2 = x^{1/2} - x^{-1}$ olur.

Kolayca gözleneceği üzere

$$c = P(x)W(z_1, z_2)(x) = x^{3/2}\left(\frac{3}{2}x^{-3/2}\right) = \frac{3}{2}$$

ve

$$g(x,s) = \begin{cases} -\dfrac{2}{3}s^{1/2}\left(x^{1/2} - x^{-1}\right), & 0 < s \le x \\ -\dfrac{2}{3}x^{1/2}\left(s^{1/2} - s^{-1}\right), & x < s < 1 \end{cases}$$

bulunur. Böylece aranan çözüm $y = y_p$, $(y_H = 0)$

$$y = -\int_0^1 g(x,s)(2s)ds$$

$$= \frac{2}{3}\left(x^{1/2} - x^{-1}\right)\int_0^x s^{1/2}(2s)ds + \frac{2}{3}x^{1/2}\int_x^1 \left(s^{1/2} - s^{-1}\right)(2s)ds$$

ve gerekli işlemler yapılırsa

$$y = \frac{4}{5}x^{1/2}(x - 1)$$

elde edilir.

## 2. YAPILAN ÇALIŞMALAR

Bu bölümde

$$y'' + (\lambda - q)y = 0 \tag{2.1}$$

denkleminin

$$\frac{y'(0,\lambda)}{y(0,\lambda)} = -tan\alpha \tag{2.2}$$

$$cos\beta y'(\pi,\lambda) + sin\beta y(\pi,\lambda) = 0 \tag{2.3}$$

$\alpha, \beta \in [0,\pi]$ sınır koşullarını sağlayan, $q \in L^1[0,\pi]$ ve $\int_0^\pi q(t)dt = 0$ (ortalama değeri sıfır) olan Green fonksiyonu genel olarak

$$G(x,y,\lambda) = \begin{cases} \dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)} + O\left(\lambda^{-k}\eta(\lambda)\right), & 0 \le x \le y \le \pi \\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)} + O\left(\lambda^{-k}\eta(\lambda)\right), & 0 \le y \le x \le \pi \end{cases}$$

şeklinde hesaplanacaktır. Bu amaçla ilk olarak $\varphi, \psi$ ve $w(\lambda)$ için asimptotik çözümler elde edilecektir. Uygulanan yöntem [1] ve [2] çalışmasındaki yöntemlere benzerdir.

Bu tür çalışmalardan bazıları sınır değerlerinde özdeğer parametresi içermektedir. Özdeğer parametresi tek sınır koşulunda olabileceği gibi her iki sınır koşulunda da bulanabilir. [1] çalışmasında $\lambda -$bağımlı sol sınır koşulu quadratik, sağ sınır koşulu ise sadece bağımlı ve türevini içermektedir. [2] çalışmasında ise $\lambda$ parametresi her iki sınır koşullarında da bulunmaktadır. [3] çalışmasında da $\lambda$ parametresi her iki sınır koşulunda lineer olarak yer almaktadır.

Öncelikle (2.1) denklemi uygun dönüşümle

$$v' = -\lambda + q - v^2 \tag{2.4}$$

dönüştürülür [4].

$S(x,\lambda) \coloneqq Rev(x,\lambda)$ ve $T(x,\lambda) \coloneqq Imv(x,\lambda)$ olmak üzere

$$F(x,\lambda) := \begin{cases} \left|\int_x^{\pi} e^{2i\lambda^{(1/2)t}} q(t)dt\right| / \int_x^{\pi} |q(t)dt|, & \int_x^{\pi} |q(t)|dt \neq 0 \\ 0, & \int_x^{\pi} |q(t)|dt = 0 \end{cases}$$

olarak tanımlanırsa

$$\eta(\lambda) := \{ \sup_{0 \leq x \leq \pi} F(x,\lambda),$$

$$\eta(\lambda) \to 0, \lambda \to \infty \quad [4].$$

(2.4) için yaklaşık çözümler aşağıdaki gibidir [4] :

$$v(x,\lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(x,\lambda)$$

$$v_1' + 2i\lambda^{1/2} v_1 = q,$$

$$v_2' + 2i\lambda^{1/2} v_2 = -v_1^2,$$

ve n=3,4,… için

$$v_n' + 2i\lambda^{1/2} v_n = -\left(v_{n-1}^2 + 2v_{n-1} \sum_{m=1}^{n-2} v_m\right) \qquad (2.5).$$

**Teorem 2.1:**

$$y'' + (\lambda - q)y = 0 \qquad (2.1)$$

denkleminin

$$y(0,\lambda) = cos\alpha, \qquad y'(0,\lambda) = -sin\alpha, \quad \alpha \in [0,\pi] \qquad (2.6)$$

başlangıç koşullarını sağlayan çözümü

$$\varphi(x,\lambda) = cos\alpha \times \exp\left(\int_0^x S(t,\lambda)dt\right)$$

$$\times \left[cos\left(\int_0^x T(t,\lambda)dt\right) - \left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right] \times sin\left(\int_0^x T(t,\lambda)dt\right)\right] \qquad (2.7)$$

ile verilir. Burada $S(x,\lambda), T(x,\lambda)$ önceden tanımlandığı gibidir.

**İspat:**

(2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda) = c_1\, \mathrm{S(x,\lambda)exp}\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^x T(t,\lambda)dt\right) T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.6) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.6) ile verilen başlangıç koşullarından $Z(0,\lambda) = cos\alpha$ yerine yazılırsa

$$cos\alpha = c_1 \exp\left(\int_0^0 S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^0 T(t,\lambda)dt\right),$$

$$cos\alpha = c_1 cos c_2\,, \qquad c_1 = \frac{cos\alpha}{cos c_2} \tag{2.10}$$

$Z'(0,\lambda) = -sin\alpha$ kullanılırsa

$$-sin\alpha = c_1 S(0,\lambda) \exp\left(\int_0^0 S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^0 T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^0 S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^0 T(t,\lambda)dt\right) T(0,\lambda),$$

$$-sin\alpha = c_1 S(0,\lambda) cos c_2 - c_1 sin c_2 T(0,\lambda) \tag{2.11}$$

bulunur. (2.10) eşitliğinden bulunan $c_1$ değeri (2.11)'de yerine yazılırsa;

$$-sin\alpha = \frac{cos\alpha}{cos c_2} S(0,\lambda) cos c_2 - \frac{cos\alpha}{cos c_2} sin c_2 T(0,\lambda),$$

$$-sin\alpha = cos\alpha \times S(0,\lambda) - cos\alpha \times tan c_2 T(0,\lambda),$$

$$-sin\alpha = cos\alpha[S(0,\lambda) - tanc_2 T(0,\lambda)]$$

$$tanc_2 = \frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)} \quad \Rightarrow \quad c_2 = tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right] \tag{2.12}$$

(2.10) eşitliği kullanılırsa;

$$c_1 = \frac{cos\alpha}{cos\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right]\right)} \tag{2.13}$$

bulunur. Son olarak (2.12) ve (2.13) (2.7)'de yerine yazılırsa (2.6) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x,\lambda)$ denilirse

$$\varphi(x,\lambda) = \frac{cos\alpha}{cos\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right]\right)} \times \exp\left(\int_0^x S(t,\lambda)dt\right)$$

$$\times\, cos\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right] + \int_0^x T(t,\lambda)dt\right) \tag{2.14}$$

bulunur.

$\cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$ kullanılırsa

$$cos\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right] + \int_0^x T(t,\lambda)dt\right)$$

$$= cos\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right]\right) \times cos\left(\int_0^x T(t,\lambda)dt\right)$$

$$-sin\left(tan^{-1}\left[\frac{S(0,\lambda) + tan\alpha}{T(0,\lambda)}\right]\right) \times sin\left(\int_0^x T(t,\lambda)dt\right)$$

(2.14)'de yerine yazılırsa ve gerekli işlemler yapılırsa

$$\varphi(x,\lambda) = cos\alpha \times \exp\left(\int_0^x S(t,\lambda)dt\right)$$

$$\times\left[\begin{array}{l}\dfrac{\cos\left(tan^{-1}\left[\dfrac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)\times\cos\left(\int_0^x T(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\dfrac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)}\\ -\dfrac{sin\left(tan^{-1}\left[\dfrac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)\times sin\left(\int_0^x T(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\dfrac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)}\end{array}\right]$$

düzenlenirse;

$$\varphi(x,\lambda)=cos\alpha\times\exp\left(\int_0^x S(t,\lambda)dt\right)$$

$$\times\left[cos\left(\int_0^x T(t,\lambda)dt\right)-\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\times sin\left(\int_0^x T(t,\lambda)dt\right)\right]\quad(2.7)$$

elde edilir.

Aşağıdaki notasyonları kullanırsak

$$E_0(x,\lambda)\coloneqq\exp\left(\int_0^x S(t,\lambda)dt\right),$$

$$C_0(x,\lambda)\coloneqq cos\left(\int_0^x T(t,\lambda)dt\right),$$

$$S_0(x,\lambda)\coloneqq sin\left(\int_0^x T(t,\lambda)dt\right)$$

$\varphi(x,\lambda)$ eşitliğimiz ;

$$\varphi(x,\lambda)=cos\alpha E_0(x,\lambda)\left[C_0(x,\lambda)-\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]S_0(x,\lambda)\right],$$

$$\varphi(x,\lambda)=cos\alpha E_0(x,\lambda)C_0(x,\lambda)-cos\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}$$

$$-sin\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{1}{T(0,\lambda)}\quad(2.15)$$

şeklinde elde edilir. Son eşitlikteki $E_0(x,\lambda), C_0(x,\lambda), S_0(x,\lambda), S(0,\lambda)$ ve $T(0,\lambda)$değerleri daha sonra hesaplanacaktır.

**Teorem 2.2:**

$$y'' + (\lambda - q)y = 0 \tag{2.1}$$

denkleminin

$$y(\pi,\lambda) = cos\beta, \qquad y'(\pi,\lambda) = -sin\beta, \quad \beta \in [0,\pi] \tag{2.16}$$

başlangıç koşullarını sağlayan çözümü

$$\psi(x,\lambda) = cos\beta \exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$\times \left[cos\left(\int_x^\pi T(t,\lambda)dt\right) + \left[\frac{S(\pi,\lambda) + tan\beta}{T(\pi,\lambda)}\right] \times sin\left(\int_x^\pi T(t,\lambda)dt\right)\right] \tag{2.17}$$

ile verilir. Burada $S(x,\lambda), T(x,\lambda)$ önceden tanılandığı gibidir.

**İspat:**

(2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda) = c_1 \mathrm{S(x,\lambda)} \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^x T(t,\lambda)dt\right) T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.16) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.16) ile verilen başlangıç koşullarından

$Z(\pi,\lambda) = cos\beta$ ve $Z'(\pi,\lambda) = -sin\beta$ yazılırsa

$$cos\beta = c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right), \tag{2.18}$$

$$-sin\beta = c_1\, \mathrm{S}(\pi,\lambda)\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right) T(\pi,\lambda) \qquad (2.19)$$

(2.18) eşitliği (2.19) eşitliğinde kullanılırsa

$$-sin\beta = \frac{cos\beta \mathrm{S}(\pi,\lambda)\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right)}{\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right)}$$

$$-\frac{cos\beta \exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right) T(\pi,\lambda)}{\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right)},$$

$$-sin\beta = cos\beta \mathrm{S}(\pi,\lambda) - cos\beta \mathrm{T}(\pi,\lambda) tan\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right),$$

$$tan\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right) = \frac{cos\beta \mathrm{S}(\pi,\lambda) + sin\beta}{cos\beta \mathrm{T}(\pi,\lambda)},$$

$$c_2 + \int_0^{\pi} T(t,\lambda)dt = tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda) + tan\beta}{\mathrm{T}(\pi,\lambda)}\right],$$

$$c_2 = tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda) + tan\beta}{\mathrm{T}(\pi,\lambda)}\right] - \int_0^{\pi} T(t,\lambda)dt \qquad (2.20)$$

bulunur. (2.20) eşitliği (2.18)'de yerine yazılırsa

$$cos\beta = c_1 \exp\left(\int_0^{\pi} S(t,\lambda)dt\right)$$

$$\times cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda) + tan\beta}{\mathrm{T}(\pi,\lambda)}\right] - \int_0^{\pi} T(t,\lambda)dt + \int_0^{\pi} T(t,\lambda)dt\right),$$

$$c_1 = \frac{cos\beta}{\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda) + tan\beta}{\mathrm{T}(\pi,\lambda)}\right]\right)} \qquad (2.21)$$

elde edilir. Son olarak (2.20) ve (2.21) eşitlikleri (2.17)'de yerine yazılırsa (2.16) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x,\lambda)$ denilirse

$$\psi(x,\lambda)=\frac{cos\beta\,\exp\left(\int_0^x S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right)cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda)+tan\beta}{\mathrm{T}(\pi,\lambda)}\right]\right)}$$

$$\times\, cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda)+tan\beta}{\mathrm{T}(\pi,\lambda)}\right]-\int_0^\pi T(t,\lambda)dt+\int_0^x T(t,\lambda)dt\right)$$

sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x,\lambda)=\frac{cos\beta\,\exp\left(-\int_x^\pi S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda)+tan\beta}{\mathrm{T}(\pi,\lambda)}\right]\right)}$$

$$\times\, cos\left(tan^{-1}\left[\frac{\mathrm{S}(\pi,\lambda)+tan\beta}{\mathrm{T}(\pi,\lambda)}\right]-\int_x^\pi T(t,\lambda)dt\right) \tag{2.22}$$

bulunur.

$\cos(\alpha-\beta)=cos\alpha cos\beta+sin\alpha sin\beta$ kullanılırsa

$$cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]-\int_x^\pi T(t,\lambda)dt\right)$$

$$=cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)\times cos\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$+sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)\times sin\left(\int_x^\pi T(t,\lambda)dt\right)$$

(2.22)'de yerine yazılırsa ve gerekli işlemler yapılırsa

$$\psi(x,\lambda)=cos\beta\times\exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$\times\left[\begin{array}{c}\dfrac{\cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)\times\cos\left(\int_x^\pi T(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)}\\ +\dfrac{sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)\times sin\left(\int_x^\pi T(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)}\end{array}\right],$$

$$= cos\beta \times \exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$\times\left[\cos\left(\int_x^\pi T(t,\lambda)dt\right)+\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]sin\left(\int_x^\pi T(t,\lambda)dt\right)\right] \qquad (2.17)$$

elde edilir.

Aşağıdaki notasyonları kullanırsak

$$E_\pi(x,\lambda) \coloneqq \exp\left(-\int_x^\pi S(t,\lambda)dt\right),$$

$$C_\pi(x,\lambda) \coloneqq cos\left(\int_x^\pi T(t,\lambda)dt\right),$$

$$S_\pi(x,\lambda) \coloneqq sin\left(\int_x^\pi T(t,\lambda)dt\right)$$

$\psi(x,\lambda)$ eşitliğimiz ;

$$\psi(x,\lambda) = cos\beta E_\pi(x,\lambda)C_\pi(x,\lambda) + cos\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}$$

$$+sin\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)} \qquad (2.23)$$

şeklinde bulunur. Son eşitlikteki $E_\pi(x,\lambda), C_\pi(x,\lambda), S_\pi(x,\lambda), S(\pi,\lambda)$ ve $T(\pi,\lambda)$ değerleri daha sonra hesaplanacaktır.

Öncelikle $S(x,\lambda)$ ve $T(x,\lambda)$ değerlerini hesaplayalım. Bunun için [4][sayfa 870].

Yeterince büyük $\lambda$ için $|v_2(x,\lambda)| \leq k_2\eta(\lambda)^2$ olan bir $\{k_2\}$ reel sayı dizisi vardır. Dolayısıyla

$$v(x,\lambda) = i\lambda^{1/2} + v_1(x,\lambda) + k_2\eta(\lambda)^2$$

olur. (2.5)'den

$$v_1(x,\lambda) = -e^{-2i\lambda^{1/2}x}\int_x^\pi e^{2i\lambda^{1/2}t}q(t)dt$$

dir. Düzenlemeler yapılırsa

$$v_1(x,\lambda) = -\left[cos\left(2\lambda^{1/2}x\right) - isin\left(2\lambda^{1/2}x\right)\right]$$

$$\times \int_x^{\pi} \left[cos\left(2\lambda^{1/2}t\right) + isin\left(2\lambda^{1/2}t\right)\right] q(t)dt$$

$$= -cos\left(2\lambda^{1/2}x\right) \int_x^{\pi} cos\left(2\lambda^{1/2}t\right) q(t)dt$$

$$-icos\left(2\lambda^{1/2}x\right) \int_x^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt$$

$$+isin\left(2\lambda^{1/2}x\right) \int_x^{\pi} cos\left(2\lambda^{1/2}t\right) q(t)dt - sin\left(2\lambda^{1/2}x\right) \int_x^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt$$

elde edilir.

$$T(x,\lambda) = Imv(x,\lambda) = \lambda^{1/2} + Imv_1(x,\lambda) + O(\eta(\lambda)^2),$$

$$Imv_1(x,\lambda) = -cos\left(2\lambda^{1/2}x\right) \int_x^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt$$

$$+sin\left(2\lambda^{1/2}x\right) \int_x^{\pi} cos\left(2\lambda^{1/2}t\right) q(t)dt,$$

$$T(x,\lambda) = \lambda^{1/2} - cos\left(2\lambda^{1/2}x\right) \int_x^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt$$

$$+sin\left(2\lambda^{1/2}x\right) \int_x^{\pi} cos\left(2\lambda^{1/2}t\right) q(t)dt + O(\eta(\lambda)^2)$$

bulunur.

$$S(x,\lambda) = Rev(x,\lambda) = Rev_1(x,\lambda) + O(\eta(\lambda)^2),$$

$$Rev_1(x,\lambda) = -cos\left(2\lambda^{1/2}x\right) \int_x^{\pi} cos\left(2\lambda^{1/2}t\right) q(t)dt$$

$$-sin\left(2\lambda^{1/2}x\right) \int_x^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt,$$

$$S(x,\lambda)=-cos\left(2\lambda^{1/2}x\right)\int_x^{\pi}cos\left(2\lambda^{1/2}t\right)q(t)dt$$

$$-sin\left(2\lambda^{1/2}x\right)\int_x^{\pi}sin\left(2\lambda^{1/2}t\right)q(t)dt+O(\eta(\lambda)^2)$$

elde edilir. Burada

$$sin\xi_x=\int_x^{\pi}cos\left(2\lambda^{1/2}t\right)q(t)dt\,,\quad cos\xi_x=\int_x^{\pi}sin\left(2\lambda^{1/2}t\right)q(t)dt \tag{2.24}$$

olarak alınırsa

$$T(x,\lambda)=\lambda^{1/2}-\left[cos\left(2\lambda^{1/2}x\right)cos\xi_x-sin\left(2\lambda^{1/2}x\right)sin\xi_x\right]+O(\eta(\lambda)^2)$$

bulunur ve $\cos(\alpha+\beta)=cos\alpha cos\beta-sin\alpha sin\beta$ eşitliği kullanılırsa

$$T(x,\lambda)=\lambda^{1/2}-cos\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2) \tag{2.25}$$

olur.

$$S(x,\lambda)=-\left[cos\left(2\lambda^{1/2}x\right)sin\xi_x+sin\left(2\lambda^{1/2}x\right)cos\xi_x\right]+O(\eta(\lambda)^2)$$

bulunur ve $\sin(\alpha+\beta)=sin\alpha cos\beta+cos\alpha sin\beta$ bağıntısı ile

$$S(x,\lambda)=-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2) \tag{2.26}$$

elde edilir.

$$\frac{S(x,\lambda)}{T(x,\lambda)}=\frac{-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)}{\lambda^{1/2}-cos(2\lambda^{1/2}x+\xi_x)+O(\eta(\lambda)^2)}$$

$$=-\left[\frac{sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)}{\lambda^{1/2}}\right]$$

$$\times\left[1+\lambda^{-1/2}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=-\lambda^{-1/2}sin\left(2\lambda^{1/2}x+\xi_x\right)-\lambda^{-1}sin\left(2\lambda^{1/2}x+\xi_x\right)cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$=-\lambda^{-1/2}sin\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \tag{2.27}$$

bulunur. Burada

$$sin\left(2\lambda^{1/2}x+\xi_x\right)=O\left(\eta(\lambda)\right),\qquad cos\left(2\lambda^{1/2}x+\xi_x\right)=O\left(\eta(\lambda)\right) \tag{2.28}$$

dir[4]. İlk olarak

$$T(0,\lambda)=T(\pi,\lambda)=\lambda^{1/2}+O\left(\eta(\lambda)\right) \tag{2.29}$$

ikinci olarak

$$\frac{1}{T(0,\lambda)}=\frac{1}{T(\pi,\lambda)}=\frac{1}{\lambda^{1/2}+O\left(\eta(\lambda)\right)}=\frac{1}{\lambda^{1/2}\left[1+O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]}$$

$$=\frac{1}{\lambda^{1/2}}\times\left[1+O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]=\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right), \tag{2.30}$$

üçüncü olarak

$$S(0,\lambda)=S(\pi,\lambda)=O\left(\eta(\lambda)\right) \tag{2.31}$$

ve son olarak

$$\frac{S(0,\lambda)}{T(0,\lambda)}=\frac{S(\pi,\lambda)}{T(\pi,\lambda)}=\frac{O\left(\eta(\lambda)\right)}{\lambda^{1/2}+O\left(\eta(\lambda)\right)}=\frac{O\left(\eta(\lambda)\right)}{\lambda^{1/2}\left[1+O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]}$$

$$=\frac{O\left(\eta(\lambda)\right)}{\lambda^{1/2}}\times\left[1+O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]=O\left(\lambda^{-1/2}\eta(\lambda)\right)+O\left(\lambda^{-1}\eta(\lambda)\right)$$

$$=O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.32}$$

elde edilir.

**Teorem 2.3:** $\lambda\to\infty$ için

**a)**

$$E_0(x,\lambda)=\exp\left(\int_0^x S(t,\lambda)dt\right)=1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.33}$$

**b)**

$$C_0(x,\lambda)=cos\left(\int_0^x T(t,\lambda)dt\right)$$

$$=cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.34}$$

**c)**

$$S_0(x,\lambda)=sin\left(\int_0^x T(t,\lambda)dt\right)$$

$$=sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.35}$$

**d)**

$$E_\pi(x,\lambda)=\exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$=1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.36}$$

**e)**

$$C_\pi(x,\lambda)=cos\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$=cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.37}$$

**f)**

$$S_\pi(x,\lambda)=sin\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$=sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right), \tag{2.38}$$

ile verilir.

**İspat:**

**a)** Öncelikle $\int_0^x S(t,\lambda)dt$ integralini hesaplayalım.

$$\int_0^x S(t,\lambda)dt = -\int_0^x \cos\left(2\lambda^{1/2}t\right)\left(\int_t^\pi \cos\left(2\lambda^{1/2}s\right) q(s)ds\right)dt$$

$$-\int_0^x \sin\left(2\lambda^{1/2}t\right)\left(\int_t^\pi \sin\left(2\lambda^{1/2}s\right) q(s)ds\right)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$\coloneqq I_1 + I_2$$

$I_1$ için aşağıdaki işlemler yapılırsa

$$I_1 = -\int_0^x \cos\left(2\lambda^{1/2}t\right)\left(\int_t^\pi \cos\left(2\lambda^{1/2}s\right) q(s)ds\right)dt$$

$$dv = \cos\left(2\lambda^{1/2}t\right)dt \qquad v = \frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}t\right)$$

$$u = \int_t^\pi \cos\left(2\lambda^{1/2}s\right) q(s)ds \qquad du = 0 - \cos\left(2\lambda^{1/2}t\right)q(t)dt$$

$$I_1 = -\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}t\right)\int_t^\pi \cos\left(2\lambda^{1/2}s\right) q(s)ds\,\Big|_0^x$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^x \sin\left(2\lambda^{1/2}t\right)\cos\left(2\lambda^{1/2}t\right)q(t)dt.$$

elde edilir. Benzer şekilde

$$I_2 = -\int_0^x \sin\left(2\lambda^{1/2}t\right)\left(\int_t^\pi \sin\left(2\lambda^{1/2}s\right) q(s)ds\right)dt$$

$$dv = \sin\left(2\lambda^{1/2}t\right)dt \qquad v = -\frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}t\right)$$

$$u = \int_t^\pi \sin\left(2\lambda^{1/2}s\right) q(s)ds \qquad du = 0 - \sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$I_2 = \frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}t\right)\int_t^\pi \sin\left(2\lambda^{1/2}s\right) q(s)ds\,\Big|_0^x$$

$$+\frac{1}{2\lambda^{1/2}}\int_0^x \sin\left(2\lambda^{1/2}t\right)\cos\left(2\lambda^{1/2}t\right)q(t)dt$$

bulunur. $I_1$ ve $I_2$'yi $\int_0^x S(t,\lambda)dt$ integralinde yerine yazarsak

$$\int_0^x S(t,\lambda)dt = -\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}t\right)\int_t^\pi \cos\left(2\lambda^{1/2}s\right)q(s)ds\Big|_0^x$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^x \sin\left(2\lambda^{1/2}t\right)\cos\left(2\lambda^{1/2}t\right)q(t)dt$$

$$+\frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}t\right)\int_t^\pi \sin\left(2\lambda^{1/2}s\right)q(s)ds\Big|_0^x$$

$$+\frac{1}{2\lambda^{1/2}}\int_0^x \sin\left(2\lambda^{1/2}t\right)\cos\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$=\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x\right)\int_x^\pi \sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^\pi \sin\left(2\lambda^{1/2}x\right)q(x)\,dx$$

$$-\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}x\right)\int_x^\pi \cos\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

olarak elde edilir ve (2.24) kullanılırsa

$$\int_0^x S(t,\lambda)dt = \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x\right)cos\xi_x - \frac{1}{2\lambda^{1/2}}sin\left(2\lambda^{1/2}x\right)sin\xi_x$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^\pi \sin\left(2\lambda^{1/2}x\right)q(x)\,dx + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$\int_0^x S(t,\lambda)dt = \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi \sin\left(2\lambda^{1/2}x\right)q(x)\,dx$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

bulunur ve Taylor seri açılımı ile

$$E_0(x,\lambda) = \exp\left(\int_0^x S(t,\lambda)dt\right) = 1 + \frac{1}{2\lambda^{1/2}} cos\big(2\lambda^{1/2}x + \xi_x\big)$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\big(2\lambda^{1/2}x\big)q(x)dt + O\big(\lambda^{-1/2}\eta(\lambda)^2\big) \qquad (2.33)$$

elde edilir.

**b)** Öncelikle $\int_0^x T(t,\lambda)dt$'yı hesaplayalım.

$$\int_0^x T(t,\lambda)dt = \int_0^x \lambda^{1/2}dt + \int_0^x \sin\big(2\lambda^{1/2}t\big)\left(\int_t^\pi \cos\big(2\lambda^{1/2}s\big)q(s)ds\right)dt$$

$$-\int_0^x \cos\big(2\lambda^{1/2}t\big)\left(\int_t^\pi \sin\big(2\lambda^{1/2}s\big)q(s)ds\right)dt + O\big(\lambda^{-1/2}\eta(\lambda)^2\big)$$

$$:= I_1 + I_2$$

$I_1$ integrali

$$I_1 = \int_0^x \sin\big(2\lambda^{1/2}t\big)\left(\int_t^\pi \cos\big(2\lambda^{1/2}s\big)q(s)ds\right)dt$$

$$dv = \sin\big(2\lambda^{1/2}t\big)dt \qquad v = -\frac{1}{2\lambda^{1/2}}\cos\big(2\lambda^{1/2}t\big)$$

$$u = \int_t^\pi \cos\big(2\lambda^{1/2}s\big)\,q(s)ds \qquad du = 0 - \cos\big(2\lambda^{1/2}t\big)q(t)dt$$

$$I_1 = -\frac{1}{2\lambda^{1/2}}\cos\big(2\lambda^{1/2}t\big)\int_t^\pi \cos\big(2\lambda^{1/2}s\big)q(s)ds\,\Big|_0^x$$

$$-\frac{1}{2\lambda^{1/2}}\int_0^x \cos\big(2\lambda^{1/2}t\big)\cos\big(2\lambda^{1/2}t\big)\,q(t)dt$$

şeklinde hesaplanır. Benzer şekilde $I_2$ integrali de

$$I_2 = -\int_0^x \cos\big(2\lambda^{1/2}t\big)\left(\int_t^\pi \sin\big(2\lambda^{1/2}s\big)q(s)ds\right)dt$$

$$dv = \cos\big(2\lambda^{1/2}t\big)dt \qquad v = \frac{1}{2\lambda^{1/2}}\sin\big(2\lambda^{1/2}t\big)$$

$$u = \int_t^{\pi} \sin\left(2\lambda^{1/2}s\right) q(s)ds \qquad du = 0 - \sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$I_2 = -\frac{1}{2\lambda^{1/2}} \sin\left(2\lambda^{1/2}t\right) \int_t^{\pi} \sin\left(2\lambda^{1/2}s\right) q(s)ds \,\Big|_0^x$$

$$-\frac{1}{2\lambda^{1/2}} \int_0^x \sin\left(2\lambda^{1/2}t\right) \sin\left(2\lambda^{1/2}t\right) q(t)dt$$

bulunur. Böylece

$$I_1 + I_2 = -\frac{1}{2\lambda^{1/2}} \cos\left(2\lambda^{1/2}x\right) \int_x^{\pi} \cos\left(2\lambda^{1/2}t\right)q(t)dt$$

$$+\frac{1}{2\lambda^{1/2}} \int_0^{\pi} cos\left(2\lambda^{1/2}x\right) q(x)dx$$

$$-\frac{1}{2\lambda^{1/2}} \sin\left(2\lambda^{1/2}x\right) \int_x^{\pi} \sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$-\frac{1}{2\lambda^{1/2}} \int_0^x cos^2\left(2\lambda^{1/2}t\right)q(t)dt$$

$$-\frac{1}{2\lambda^{1/2}} \int_0^x sin^2\left(2\lambda^{1/2}t\right)q(t)dt$$

elde edilir. (2.24) kullanılırsa ve gerekli düzenlemeler yapılırsa

$$I_1 + I_2 = -\frac{1}{2\lambda^{1/2}} \left[cos\left(2\lambda^{1/2}x\right)sin\xi_x + sin\left(2\lambda^{1/2}x\right)cos\xi_x\right]$$

$$+\frac{1}{2\lambda^{1/2}} \int_0^{\pi} cos\left(2\lambda^{1/2}x\right) q(x)dx$$

$$-\frac{1}{2\lambda^{1/2}} \left[\int_0^x \left[cos^2\left(2\lambda^{1/2}t\right) + sin^2\left(2\lambda^{1/2}t\right)\right]q(t)dt\right]$$

$$= -\frac{1}{2\lambda^{1/2}} sin\left(2\lambda^{1/2}x + \xi_x\right) + \frac{1}{2\lambda^{1/2}} \int_0^{\pi} cos\left(2\lambda^{1/2}x\right) q(x)dx$$

$$-\frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt$$

olur, buradan

$$\int_0^x T(t,\lambda)dt = \lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} sin(2\lambda^{1/2}x + \xi_x)$$

$$+\frac{1}{2\lambda^{1/2}}\int_0^\pi cos(2\lambda^{1/2}x)\, q(x)dx - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt$$

$$+O(\lambda^{-1/2}\eta(\lambda)^2)$$

bulunur ve

$$C_0(x,\lambda) = cos\left(\int_0^x T(t,\lambda)dt\right)$$

$$= cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \qquad (2.34)$$

elde edilir.

**c)** $\int_0^x T(t,\lambda)dt$ b)'deki gibi elde edilir. Buradan hareketle

$$S_0(x,\lambda) = sin\left(\int_0^x T(t,\lambda)dt\right)$$

$$= sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2) \qquad (2.35)$$

bulunur.

**d)** Öncelikle $-\int_x^\pi S(t,\lambda)dt$ integralini hesaplayalım.

$$-\int_x^\pi S(t,\lambda)dt = \int_x^\pi \cos(2\lambda^{1/2}t)\left(\int_t^\pi \cos(2\lambda^{1/2}s)\, q(s)ds\right)dt$$

$$+\int_x^\pi \sin(2\lambda^{1/2}t)\left(\int_t^\pi \sin(2\lambda^{1/2}s)\, q(s)ds\right)dt + O(\lambda^{-1/2}\eta(\lambda)^2)$$

$$\coloneqq I_1 + I_2$$

$I_1$ ve $I_2$'yı bulalım. İlk olarak

$$I_1 = \int_x^\pi \cos(2\lambda^{1/2}t)\left(\int_t^\pi \cos(2\lambda^{1/2}s)\, q(s)ds\right)dt$$

$$dv = \cos(2\lambda^{1/2}t)dt \qquad v = \frac{1}{2\lambda^{1/2}}\sin(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \cos(2\lambda^{1/2}s)\, q(s)ds \qquad du = 0 - \cos(2\lambda^{1/2}t)q(t)dt$$

$$I_1 = \frac{1}{2\lambda^{1/2}}\sin(2\lambda^{1/2}t)\int_t^\pi \cos(2\lambda^{1/2}s)\, q(s)ds\,\Big|_x^\pi$$

$$+\frac{1}{2\lambda^{1/2}}\int_x^\pi \sin(2\lambda^{1/2}t)\cos(2\lambda^{1/2}t)q(t)\,dt.$$

elde edilir. Benzer şekilde

$$I_2 = \int_x^\pi \sin(2\lambda^{1/2}t)\left(\int_t^\pi \sin(2\lambda^{1/2}s)\, q(s)ds\right)dt$$

$$dv = \sin(2\lambda^{1/2}t)dt \qquad v = -\frac{1}{2\lambda^{1/2}}\cos(2\lambda^{1/2}t)$$

$$u = \int_t^\pi \sin(2\lambda^{1/2}s)\, q(s)ds \qquad du = 0 - \sin(2\lambda^{1/2}t)q(t)dt$$

$$I_2 = -\frac{1}{2\lambda^{1/2}}\cos(2\lambda^{1/2}t)\int_t^\pi \sin(2\lambda^{1/2}s)\, q(s)ds\,\Big|_x^\pi$$

$$-\frac{1}{2\lambda^{1/2}}\int_x^\pi \sin(2\lambda^{1/2}t)\cos(2\lambda^{1/2}t)q(t)\,dt$$

bulunur. $I_1$ ve $I_2$'yi $\int_x^\pi S(t,\lambda)dt$ integralinde yerine yazarsak

$$-\int_x^\pi S(t,\lambda)dt = -\frac{1}{2\lambda^{1/2}}\sin(2\lambda^{1/2}x)\int_x^\pi \cos(2\lambda^{1/2}t)\, q(t)dt$$

$$+\frac{1}{2\lambda^{1/2}}\cos(2\lambda^{1/2}x)\int_x^\pi \sin(2\lambda^{1/2}t)\, q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

ve (2.24) kullanılırsa

$$-\int_x^{\pi} S(t,\lambda)dt = \frac{1}{2\lambda^{1/2}}\left[\cos\left(2\lambda^{1/2}x\right)cos\xi_x - \sin\left(2\lambda^{1/2}x\right)sin\xi_x\right] + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$-\int_x^{\pi} S(t,\lambda)dt = \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

elde edilir. Dolayısıyla

$$E_\pi(x,\lambda) = \exp\left(-\int_x^{\pi} S(t,\lambda)dt\right)$$

$$= 1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \tag{2.36}$$

hesaplanır.

**e)** Öncelikle $\int_x^{\pi} T(t,\lambda)dt$ integralini hesaplayalım.

$$\int_x^{\pi} T(t,\lambda)dt = \int_x^{\pi} \lambda^{1/2}dt + \int_x^{\pi} \sin\left(2\lambda^{1/2}t\right)\left(\int_t^{\pi} \cos\left(2\lambda^{1/2}s\right)q(s)ds\right)dt$$

$$-\int_x^{\pi} \cos\left(2\lambda^{1/2}t\right)\left(\int_t^{\pi} \sin\left(2\lambda^{1/2}s\right)q(s)ds\right)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$\coloneqq I_1 + I_2$$

$I_1$ integrali için

$$I_1 = \int_x^{\pi} \sin\left(2\lambda^{1/2}t\right)\left(\int_t^{\pi} \cos\left(2\lambda^{1/2}s\right)q(s)ds\right)dt$$

kısmi integrasyon yöntemi ile

$$dv = \sin\left(2\lambda^{1/2}t\right)dt \qquad v = -\frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}t\right)$$

$$u = \int_t^{\pi} \cos\left(2\lambda^{1/2}s\right) q(s)ds \qquad du = 0 - \cos\left(2\lambda^{1/2}t\right)q(t)dt$$

dönüşümleri yerine yazılır, böylece

$$I_1 = -\frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}t\right)\int_t^{\pi} \cos\left(2\lambda^{1/2}s\right) q(s)ds \,\Big|_x^{\pi}$$

$$-\frac{1}{2\lambda^{1/2}}\int_x^\pi \cos\left(2\lambda^{1/2}t\right)\cos\left(2\lambda^{1/2}t\right)q(t)dt.$$

elde edilir. Benzer şekilde $I_2$ integralinde de

$$I_2=-\int_x^\pi \cos\left(2\lambda^{1/2}t\right)\left(\int_t^\pi \sin\left(2\lambda^{1/2}s\right)q(s)ds\right)dt$$

$$dv=\cos\left(2\lambda^{1/2}t\right)dt \qquad v=\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}t\right)$$

$$u=\int_t^\pi \sin\left(2\lambda^{1/2}s\right)q(s)ds \qquad du=0-\sin\left(2\lambda^{1/2}t\right)q(t)dt$$

dönüşümleri ile

$$I_2=-\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}t\right)\int_t^\pi \sin\left(2\lambda^{1/2}s\right)q(s)ds\,\Big|_x^\pi$$

$$-\frac{1}{2\lambda^{1/2}}\int_x^\pi \sin\left(2\lambda^{1/2}t\right)\sin\left(2\lambda^{1/2}t\right)q(t)dt$$

olarak bulunur. Böylece

$$I_1+I_2=\frac{1}{2\lambda^{1/2}}\cos\left(2\lambda^{1/2}x\right)\int_x^\pi \cos\left(2\lambda^{1/2}t\right)q(t)dt$$

$$+\frac{1}{2\lambda^{1/2}}\sin\left(2\lambda^{1/2}x\right)\int_x^\pi \sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$-\frac{1}{2\lambda^{1/2}}\int_x^\pi \left[cos^2\left(2\lambda^{1/2}t\right)+sin^2\left(2\lambda^{1/2}t\right)q(t)dt\right]$$

olur. (2.24) kullanılırsa ve gerekli düzenlemeler yapılırsa

$$I_1+I_2=\frac{1}{2\lambda^{1/2}}\left[cos\left(2\lambda^{1/2}x\right)sin\xi_x+sin\left(2\lambda^{1/2}x\right)cos\xi_x\right]-\frac{1}{2\lambda^{1/2}}\int_x^\pi \mathrm{q(t)}\,dt,$$

bulduğumuz eşitliği $\int_x^\pi T(t,\lambda)dt$ integralinde yerine yazarsak

$$\int_x^\pi T(t,\lambda)dt=\lambda^{1/2}(\pi-x)+\frac{1}{2\lambda^{1/2}}sin\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_x^\pi \mathrm{q(t)}\,dt$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

bulunur ve dolayısıyla

$$C_\pi(x,\lambda) = cos\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$= cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \qquad (2.37)$$

elde edilir.

**f)** $\int_x^\pi T(t,\lambda)dt$ e)'deki gibidir. Buradan hareketle

$$S_\pi(x,\lambda) = sin\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$= sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \qquad (2.38)$$

elde edilir.

**Teorem 2.4:** $\lambda \to \infty$ için

**i)**

$$y'' + (\lambda - q)y = 0 \qquad (2.1)$$

denkleminin

$$y(0,\lambda) = cos\alpha, \qquad y'(0,\lambda) = -sin\alpha, \quad \alpha \in [0,\pi] \qquad (2.6)$$

başlangıç koşullarını sağlayan çözümü

$$\varphi(x,\lambda) = cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\lambda^{-1/2}sin\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.39),$$

**ii)**

$$y'' + (\lambda - q)y = 0 \tag{2.1}$$

denkleminin

$$y(\pi,\lambda) = cos\beta, \qquad y'(\pi,\lambda) = -sin\beta, \quad \beta \in [0,\pi] \tag{2.16}$$

başlangıç koşullarını sağlayan çözümü

$$\psi(x,\lambda) = cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.40}$$

ile verilir.

**İspat:**

**i)** Bulduğumuz (2.30),(2.32),(2.33),(2.34) ve (2.35) eşitliklerini (2.15)'de yerine yazalım.

$$\varphi(x,\lambda) = cos\alpha E_0(x,\lambda)C_0(x,\lambda) - cos\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}$$

$$-sin\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{1}{T(0,\lambda)}$$

$$\coloneqq I_1 - I_2 - I_3 \tag{2.15$a$}$$

Öncelikle $I_1$'i hesaplayalım.

$$I_1 = cos\alpha E_0(x,\lambda)C_0(x,\lambda)$$

$$= cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\Big] \times \left[cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$= cos\alpha \times \left[\begin{array}{c} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \end{array}\right]$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

Böylece

$$I_1 = cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right).$$

şeklinde hesaplanır. Benzer şekilde

$$I_2 = cos\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}$$

$$= cos\alpha \times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\begin{array}{c} 1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) \\ -\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \end{array}\right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$= cos\alpha \times \left[\begin{array}{c} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \end{array}\right]$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

işlemleri ile

$$I_2 = cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

bulunur ve son olarak

$$I_3 = sin\alpha E_0(x,\lambda)S_0(x,\lambda)\frac{1}{T(0,\lambda)}$$

$$= sin\alpha \times \begin{bmatrix} 1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) \\ -\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \end{bmatrix}$$

$$\times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{-1/2} + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$= sin\alpha \times \begin{bmatrix} \lambda^{-1/2}sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +\frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\frac{\lambda^{-1}}{2}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \end{bmatrix}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3 = sin\alpha \times \lambda^{-1/2}sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Bu değerler (2.15a) eşitliğinde yerine yazılırsa

$$\varphi(x,\lambda) = cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\lambda^{-1/2}sin\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.39)$$

olarak bulunur.

**ii)** Bulduğumuz (2.30),(2.32),(2.36),(2.37) ve (2.38) eşitliklerini (2.23)'de yerine yazalım.

$$\psi(x,\lambda)=cos\beta E_\pi(x,\lambda)C_\pi(x,\lambda)+cos\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}$$

$$+sin\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}$$

$$:=I_1+I_2+I_3 \qquad (2.23a)$$

Öncelikle $I_1$'i hesaplayalım.

$$I_1=cos\beta E_\pi(x,\lambda)C_\pi(x,\lambda)$$

$$=cos\beta\times\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=cos\beta\times\left[\begin{array}{c}cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\\+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\end{array}\right]$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1=cos\beta\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right).$$

şeklinde hesaplanır. Benzer şekilde

$$I_2=cos\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}$$

$$=cos\beta\times\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$= cos\beta \times \begin{bmatrix} sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \end{bmatrix}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2 = cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

elde edilir ve son olarak

$$I_3 = sin\beta E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}$$

$$= sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{-1/2} + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$= sin\beta \times \begin{bmatrix} \lambda^{-1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \end{bmatrix}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3 = \lambda^{-1/2}sin\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. Bu değerler (2.23a) eşitliğinde yerine yazılırsa

$$\psi(x,\lambda) = cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{-1/2} sin\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.40)$$

olarak hesaplanır.

Şimdi $G(x, y, \lambda)$ Green fonksiyonunu elde etmek için gerekli olan

$$w(\lambda) \coloneqq w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

değerini hesaplayalım.

**Teorem 2.5:** $\lambda \to \infty$ için (2.6) ve (2.16) koşulları altında

$$w(\lambda) = w_x(\varphi, \psi) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda)$$

$$= 2\lambda^{1/2} cos\alpha \times cos\beta \times sin(\lambda^{1/2}\pi) + \lambda^{-1/2} sin\alpha \times sin\beta \times sin(\lambda^{1/2}\pi)$$

$$+cos\alpha \times sin\beta \times [sin(\lambda^{1/2}\pi) - cos(\lambda^{1/2}\pi)]$$

$$+sin\alpha \times cos\beta \times [sin(\lambda^{1/2}\pi) + cos(\lambda^{1/2}\pi)] + O(\lambda^{-1}\eta(\lambda)) \quad (2.41)$$

ile verilir.

**İspat:**

$\varphi'(x, \lambda)$'yı bulmak için (2.12) ve (2.13) değerlerini (2.9) eşitliğinde yerine yazalım.

$$Z'(x, \lambda) = \frac{cos\alpha. S(x, \lambda). exp\left(\int_0^x S(t, \lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(0, \lambda) + tan\alpha}{T(0, \lambda)}\right]\right)}$$

$$\times cos\left[tan^{-1}\left[\frac{S(0, \lambda) + tan\alpha}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda)dt\right]$$

$$-\frac{cos\alpha. exp\left(\int_0^x S(t, \lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(0, \lambda) + tan\alpha}{T(0, \lambda)}\right]\right)}$$

$$\times sin\left[tan^{-1}\left[\frac{S(0, \lambda) + tan\alpha}{T(0, \lambda)}\right] + \int_0^x T(t, \lambda)dt\right] T(x, \lambda)$$

$$= \frac{cos\alpha . S(x,\lambda) . E_0(x,\lambda)}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)}$$

$$\times \left[cos\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right) \times cos\left(\int_0^x T(t,\lambda)dt\right)\right.$$

$$\left. - sin\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right) \times sin\left(\int_0^x T(t,\lambda)dt\right)\right]$$

$$- \frac{cos\alpha E_0(x,\lambda)}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)}$$

$$\times \left[sin\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right) \times cos\left(\int_0^x T(t,\lambda)dt\right)\right.$$

$$+ cos\left(tan^{-1}\left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]\right)$$

$$\left. \times sin\left(\int_0^x T(t,\lambda)dt\right)\right] \times T(x,\lambda)$$

Notasyonlarla ifade edersek

$$Z'(x,\lambda) = cos\alpha \times S(x,\lambda)E_0(x,\lambda)C_0(x,\lambda)$$

$$- cos\alpha \times S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda) \times \left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right]$$

$$- cos\alpha \times E_0(x,\lambda)C_0(x,\lambda) \times \left[\frac{S(0,\lambda)+tan\alpha}{T(0,\lambda)}\right] T(x,\lambda)$$

$$+ cos\alpha \times E_0(x,\lambda)S_0(x,\lambda)T(x,\lambda)$$

$$Z'(x,\lambda) = cos\alpha \times S(x,\lambda)E_0(x,\lambda)C_0(x,\lambda) - cos\alpha \times S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}$$

$$- sin\alpha \times S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda)\frac{1}{T(0,\lambda)}$$

$$- cos\alpha E_0(x,\lambda)C_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}T(x,\lambda)$$

$$-sin\alpha E_0(x,\lambda)C_0(x,\lambda)\frac{1}{T(0,\lambda)}T(x,\lambda)-cos\alpha E_0(x,\lambda)S_0(x,\lambda)T(x,\lambda)$$

$$:=I_1-I_2-I_3-I_4-I_5-I_6 \tag{2.9a}$$

şeklinde olur. Öncelikle $I_1$ eşitliğinde $S(x,\lambda)$, $E_0(x,\lambda)$ ve $C_0(x,\lambda)$ eşitlikleri yerine yazılırsa

$$I_1=cos\alpha\times\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[-sin\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\eta(\lambda)^2\right)\right]$$

olur ve gerekli hesaplamalardan sonra

$$I_1=-cos\alpha\times\left[sin\left(2\lambda^{1/2}x+\xi_x\right)\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right.$$

$$+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\left.\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\times sin\left(2\lambda^{1/2}x+\xi_x\right)\right]$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)$$

$$I_1=O\left(\lambda^{-1/2}\eta(\lambda)^2\right).$$

olarak elde edilir. $I_2$ eşitliğinde $S(x,\lambda)$, $E_0(x,\lambda)$, $S_0(x,\lambda)$ ve $\frac{S(0,\lambda)}{T(0,\lambda)}$ yerine yazılır, böylece

$$I_2=\ cos\alpha\times\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$= \ cos\alpha \times \left[-sin\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right.$$

$$-\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times\, sin\left(2\lambda^{1/2}x+\xi_x\right)+\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\left.\times\, sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\times sin\left(2\lambda^{1/2}x+\xi_x\right)\right]+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

hesaplamalarından sonra

$$I_2 = \ O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanır. $I_3$ eşitliği için

$$I_3 = sin\alpha S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda)\frac{1}{T(0,\lambda)}$$

$$= sin\alpha \times \left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]\times\left[\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

gerekli hesaplamalardan sonra

$$= sin\alpha \times \left[-\lambda^{-1/2}sin\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right.$$

$$-\frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times\, sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{\lambda^{-1}}{2}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\times\ sin\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\Bigg]+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3=O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. Benzer şekilde $I_4$ eşitliği için

$$I_4=cos\alpha E_0(x,\lambda)C_0(x,\lambda)\frac{S(0,\lambda)}{T(0,\lambda)}T(x,\lambda)$$

$$=cos\alpha\times\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[\lambda^{1/2}-cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\eta(\lambda)^2\right)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$=cos\alpha\times\left[\lambda^{1/2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right.$$

$$+\frac{1}{2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\times cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$-\frac{1}{2}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\times cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$-\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)\times cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\left.\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\times cos\left(2\lambda^{1/2}x+\xi_x\right)\right]+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

yapılan işlemlerden sonra

$$I_4 = -\lambda^{1/2} cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde elde edilir ve $I_5$ eşitliğinde

$$I_5 = sin\alpha E_0(x,\lambda)C_0(x,\lambda)\frac{1}{T(0,\lambda)}T(x,\lambda)$$

$$= sin\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right] \times \left[cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{1/2} - cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\eta(\lambda)^2\right)\right] \times \left[\lambda^{-1/2} + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$= sin\alpha \times \left[cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right)\right.$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - \lambda^{-1/2}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$-\frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + \frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\left.\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times \int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right] + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_5 = -sin\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

olarak bulunur. Son olarak $I_6$ hesaplanırsa

$$I_6 = cos\alpha E_0(x,\lambda)S_0(x,\lambda)T(x,\lambda)$$

$$= cos\alpha \times \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt\right.$$

$$\left. + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right] \times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{1/2} - cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\eta(\lambda)^2\right)\right]$$

gerekli işlemlerden sonra

$$I_6 = cos\alpha \times \left[\lambda^{1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{1}{2} cos\left(2\lambda^{1/2}x + \xi_x\right)\right.$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \frac{1}{2} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$+ \frac{1}{2\lambda^{1/2}} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt \times cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\left. \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$I_6 = \lambda^{1/2} cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde elde edilir. Bulduğumuz bu değerler (2.9a) eşitliğinde yerine yazılırsa

$$\varphi'(x,\lambda) = -\lambda^{1/2} cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$- sin\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$-\lambda^{1/2}cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.42)$$

bulunur. Şimdi $\psi'(x,\lambda)$'yı bulmak için (2.20) ve (2.21) değerlerini (2.9)'da yerine yazılır.

$$Z'(x,\lambda) = \frac{cos\beta S(x,\lambda)exp\left(\int_0^x S(t,\lambda)dt\right)}{exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)}$$

$$\times cos\left[tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right] - \int_0^\pi T(t,\lambda)dt + \int_0^x T(t,\lambda)dt\right]$$

$$-\frac{cos\beta exp\left(\int_0^x S(t,\lambda)dt\right)}{exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right]\right)}$$

$$\times sin\left[tan^{-1}\left[\frac{S(\pi,\lambda)+tan\beta}{T(\pi,\lambda)}\right] - \int_0^\pi T(t,\lambda)dt + \int_0^x T(t,\lambda)dt\right]T(x,\lambda)$$

Sınırlar üzerinde düzenlemeler ve gerekli işlemler yapılırsa

$$Z'(x,\lambda) = cos\beta \times S(x,\lambda)E_\pi(x,\lambda)C_\pi(x,\lambda)$$

$$+cos\beta \times S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}$$

$$+sin\beta \times S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}$$

$$-cos\beta \times E_\pi(x,\lambda)C_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}T(x,\lambda)$$

$$-sin\beta \times E_\pi(x,\lambda)C_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}T(x,\lambda)$$

$$+cos\beta \times E_\pi(x,\lambda)S_\pi(x,\lambda)T(x,\lambda)$$

$$:= I_1 + I_2 + I_3 - I_4 - I_5 + I_6 \quad (2.9b)$$

olarak bulunur. $I_1$ için

$$I_1 = cos\beta S(x,\lambda)E_\pi(x,\lambda)C_\pi(x,\lambda)$$

$$= cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right]$$

gerekli hesaplamalardan sonra

$$I_1 = cos\beta \times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right.$$

$$- \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\left.\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$I_1 = O\left(\lambda^{-1/2}\eta(\lambda)^2\right).$$

şeklinde bulunur. Benzer şekilde

$$I_2 = cos\beta S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}$$

$$= cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$= cos\beta \times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right.$$

$$- \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\left.\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$I_2 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanır. $I_3$ hesaplanırsa

$$I_3 = sin\beta S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}$$

$$= sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}}cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[-sin(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)\right] \times \left[\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))\right]$$

$$= sin\beta \times \left[-\lambda^{-1/2}sin(2\lambda^{1/2}x + \xi_x) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right.$$

$$-\frac{\lambda^{-1}}{2}cos(2\lambda^{1/2}x + \xi_x) \times sin(2\lambda^{1/2}x + \xi_x)$$

$$\left.\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$I_3 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

olur. $I_4$ için

$$I_4 = cos\beta E_\pi(x,\lambda)C_\pi(x,\lambda)\frac{S(\pi,\lambda)}{T(\pi,\lambda)}T(x,\lambda)$$

$$= cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}}cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[\lambda^{1/2} - cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)\right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

işlemler yapılırsa

$$= cos\beta \times \Bigg[\lambda^{1/2}cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + \frac{1}{2}cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}}cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos(2\lambda^{1/2}x + \xi_x) \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)\Bigg]$$

$$I_4 = \lambda^{1/2}cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. $I_5$ eşitliğinde

$$I_5 = sin\beta E_\pi(x,\lambda)C_\pi(x,\lambda)\frac{1}{T(\pi,\lambda)}T(x,\lambda)$$

$$= sin\beta \times \left[1 + \frac{1}{2\lambda^{1/2}}cos(2\lambda^{1/2}x + \xi_x) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O(\lambda^{-1/2}\eta(\lambda)^2)\right]$$

$$\times \left[\lambda^{1/2} - cos(2\lambda^{1/2}x + \xi_x) + O(\eta(\lambda)^2)\right] \times \left[\lambda^{-1/2} + O(\lambda^{-1}\eta(\lambda))\right]$$

$$= sin\beta \times \Bigg[cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}}cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \lambda^{-1/2}cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \frac{\lambda^{-1}}{2}cos(2\lambda^{1/2}x + \xi_x)$$

$$\times cos(2\lambda^{1/2}x + \xi_x) \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)\Big]$$

$$I_5 = sin\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

olarak bulunur ve son olarak

$$I_6 = cos\beta E_\pi(x,\lambda)S_\pi(x,\lambda)T(x,\lambda)$$

$$= cos\beta \times \left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{1/2} - cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\eta(\lambda)^2\right)\right]$$

$$= cos\beta \times \Bigg[\lambda^{1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + \frac{1}{2}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)\Big]$$

$$I_6 = \lambda^{1/2}cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Bulunan eşitlikler (2.9b)'de yerine yazılırsa

$$\psi'(x,\lambda) = -\lambda^{1/2}cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$-sin\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.43}$$

olarak bulunur.

Bulunan (2.39),(2.40) ve (2.42),(2.43) eşitlikleri $w(\lambda)$'da yerine yazılırsa

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= \begin{bmatrix} cos\alpha \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -cos\alpha \\ \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{-1/2}sin\alpha \\ \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} -\lambda^{1/2}cos\beta \\ \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ -sin\beta \\ \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\lambda^{1/2}cos\beta \\ \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

$$- \begin{bmatrix} -\lambda^{1/2}cos\alpha \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -sin\alpha \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{1/2}cos\alpha \\ \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} cos\beta \\ \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +cos\beta \\ \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\lambda^{-1/2}sin\beta \\ sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

olur, çarpma işlemleri yapılırsa

$$w(\lambda) = -\lambda^{1/2}cos\beta \times cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$-cos\alpha \times sin\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\alpha \times cos\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\alpha \times sin\beta \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$-\lambda^{1/2}cos\beta \times cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\alpha \times sin\beta \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$-sin\alpha \times cos\beta \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\alpha \times sin\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\alpha \times sin\beta \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\alpha \times sin\beta \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$
$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\big(\lambda^{-1}\eta(\lambda)\big)$$

şeklinde elde edilir. Son eşitlikteki terimleri sırası ile 1,2, ... ,18 olarak isimlendirerek

$(1-17),(2-18),(3-4),(5-12),(6-10),(7-14),(8-15),(9-13),(11-16)$

eşleştirirsek kosinüs ve sinüs açılımlarını kullanırsak

$$w(\lambda) = -\lambda^{1/2}cos\beta \times cos\alpha$$
$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\alpha \times cos\beta$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+cos\alpha \times sin\beta$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$-cos\alpha \times sin\beta$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\alpha \times sin\beta$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt + \lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1}\eta(\lambda)\right)$$

elde edilir, integral sınırları üzerinde de düzenlemeler yapılırsa

$$w(\lambda) = \lambda^{1/2}cos\alpha \times cos\beta \times sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$+cos\alpha \times sin\beta \times sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$-cos\alpha \times sin\beta \times cos\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta \times sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\alpha \times sin\beta \times sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$+\lambda^{1/2}cos\beta \times cos\alpha \times sin\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$

$$+sin\alpha \times cos\beta \times cos\left(\lambda^{1/2}\pi - \frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right) + O\left(\lambda^{-1}\eta(\lambda)\right)$$

olarak bulunur. $\int_0^\pi q(t)dt = 0$ olduğu kullanılırsa

$$w(\lambda) = 2\lambda^{1/2}cos\alpha \times cos\beta \times sin\left(\lambda^{1/2}\pi\right) + \lambda^{-1/2}sin\alpha \times sin\beta \times sin\left(\lambda^{1/2}\pi\right)$$

$$+cos\alpha \times sin\beta \times \left[sin\left(\lambda^{1/2}\pi\right) - cos\left(\lambda^{1/2}\pi\right)\right]$$

$$+sin\alpha \times cos\beta \times \left[sin\left(\lambda^{1/2}\pi\right) + cos\left(\lambda^{1/2}\pi\right)\right] + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (2.41)$$

şeklinde elde edilir.

**Teorem 2. 6:** $\lambda \to \infty$ ve $0 \le x \le y \le \pi$ için

$$G(x,y,\lambda) =$$

$$= \frac{\begin{bmatrix} cos\alpha \\ \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -cos\alpha \\ \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{-1/2}sin\alpha \\ \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \end{bmatrix} \times \begin{bmatrix} cos\beta \\ \times cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +cos\beta \\ \times sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +\lambda^{-1/2}sin\beta \\ \times sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}cos\alpha \times cos\beta \times sin(\lambda^{1/2}\pi) + \lambda^{-1/2}sin\alpha \times sin\beta \times sin(\lambda^{1/2}\pi) \\ +cos\alpha \times sin\beta \times [sin(\lambda^{1/2}\pi) - cos(\lambda^{1/2}\pi)] \\ +sin\alpha \times cos\beta \times [sin(\lambda^{1/2}\pi) + cos(\lambda^{1/2}\pi)] \end{bmatrix}}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right). \qquad (2.44)$$

**İspat:**

$$G(x,y,\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}+O\left(\lambda^{-k}\eta(\lambda)\right), & 0\le x\le y\le\pi\\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}+O\left(\lambda^{-k}\eta(\lambda)\right), & 0\le y\le x\le\pi\end{cases}$$

şeklinde idi. Bulduğumuz (2.39),(2.40) ve (2.41) eşitliklerini $G(x,y,\lambda)'da$ yerine yazarsak

$0\le x\le y\le\pi$ için

$$G(x,y,\lambda)=\frac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}$$

$$=\frac{\begin{bmatrix} cos\alpha \\ \times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -cos\alpha \\ \times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{-1/2}sin\alpha \\ \times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{bmatrix}\times\begin{bmatrix} cos\beta \\ \times cos\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +cos\beta \\ \times sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +\lambda^{-1/2}sin\beta \\ \times sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}cos\alpha\times cos\beta\times sin(\lambda^{1/2}\pi)+\lambda^{-1/2}sin\alpha\times sin\beta\times sin(\lambda^{1/2}\pi) \\ +cos\alpha\times sin\beta\times[sin(\lambda^{1/2}\pi)-cos(\lambda^{1/2}\pi)] \\ +sin\alpha\times cos\beta\times[sin(\lambda^{1/2}\pi)+cos(\lambda^{1/2}\pi)]+O\left(\lambda^{-1}\eta(\lambda)\right)\end{bmatrix}}$$

(2.45) elde edilir.

$$F_1(x,\lambda):=\ cos\alpha\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-cos\alpha\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\lambda^{-1/2}sin\alpha\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

ve

$$F_2(y,\lambda):=cos\beta\times cos\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)$$

$$+cos\beta \times sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)$$

$$+\lambda^{-1/2}sin\beta \times sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)$$

ve son olarak

$$F_3(\alpha,\beta) \coloneqq 2\lambda^{1/2}cos\alpha \times cos\beta \times sin(\lambda^{1/2}\pi) + \lambda^{-1/2}sin\alpha \times sin\beta \times sin(\lambda^{1/2}\pi)$$

$$+cos\alpha \times sin\beta \times [sin(\lambda^{1/2}\pi) - cos(\lambda^{1/2}\pi)]$$

$$+sin\alpha \times cos\beta \times [sin(\lambda^{1/2}\pi) + cos(\lambda^{1/2}\pi)]$$

olarak tanımlarsak

$$G(x,y,\lambda) = \frac{\left(F_1(x,\lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right)\left(F_2(y,\lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right)}{F_3(\alpha,\beta) + O(\lambda^{-1}\eta(\lambda))}$$

$$= \frac{\begin{bmatrix} F_1(x,\lambda) \times F_2(y,\lambda) + F_1(x,\lambda) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) \\ +F_2(y,\lambda) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \times O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}}{F_3(\alpha,\beta) + O(\lambda^{-1}\eta(\lambda))}$$

$$= \frac{F_1(x,\lambda) \times F_2(y,\lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha,\beta) \times [1 + O(\lambda^{-1}\eta(\lambda))]}$$

$$= \frac{F_1(x,\lambda) \times F_2(y,\lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha,\beta)} \times \left[1 + O\left(\lambda^{-3/2}\eta(\lambda)\right)\right]$$

$$= \frac{F_1(x,\lambda) \times F_2(y,\lambda) + O\left(\lambda^{-1/2}\eta(\lambda)\right)}{F_3(\alpha,\beta)}$$

$$= \frac{F_1(x,\lambda) \times F_2(y,\lambda)}{F_3(\alpha,\beta)} + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde ederiz ve (2.45) eşitliği

$G(x,y,\lambda)=$

$$=\frac{\begin{bmatrix} cos\alpha \\ \times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -cos\alpha \\ \times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{-1/2}sin\alpha \\ \times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\end{bmatrix}\times\begin{bmatrix} cos\beta \\ \times cos\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +cos\beta \\ \times sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +\lambda^{-1/2}sin\beta \\ \times sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)\end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}cos\alpha\times cos\beta\times sin(\lambda^{1/2}\pi)+\lambda^{-1/2}sin\alpha\times sin\beta\times sin(\lambda^{1/2}\pi) \\ +cos\alpha\times sin\beta\times[sin(\lambda^{1/2}\pi)-cos(\lambda^{1/2}\pi)] \\ +sin\alpha\times cos\beta\times[sin(\lambda^{1/2}\pi)+cos(\lambda^{1/2}\pi)]\end{bmatrix}}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.44}$$

bulunur. Benzer şekilde $0\le y\le x\le\pi$ içinde

$$G(x,y,\lambda)=\frac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}$$

hesaplanır.

Aşağıdaki teoremlerde Dirichlet koşulları altında Green fonksiyonu elde edilecektir.

**Teorem 2.7:** $y''+(\lambda-q)y=0$ (2.1)

denkleminin

$$y(0,\lambda)=cos\alpha,\qquad y'(0,\lambda)=-sin\alpha,\quad \alpha=\frac{\pi}{2} \tag{2.46}$$

başlangıç koşullarını sağlayan çözümü

**a)**

$$\varphi(x,\lambda)=-\lambda^{-1/2}sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.47}$$

**b)**

$$\varphi(x,n)=-\frac{1}{(n+1)}\times sin\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$+O\big(n^{-2}\eta(n)\big) \tag{2.48}$$

olarak hesaplanır.

**İspat:**

**a)** (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda)=c_1\exp\left(\int_0^x S(t,\lambda)dt\right)\times cos\left(c_2+\int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda)=c_1\,\mathrm{S}(\mathrm{x},\lambda)\exp\left(\int_0^x S(t,\lambda)dt\right)\times cos\left(c_2+\int_0^x T(t,\lambda)dt\right)$$

$$-c_1\exp\left(\int_0^x S(t,\lambda)dt\right)\times sin\left(c_2+\int_0^x T(t,\lambda)dt\right)T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.46) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.46) ile verilen başlangıç koşullarından $Z(0,\lambda)=cos\alpha$ yerine yazılırsa

$$cos\alpha=c_1\exp\left(\int_0^0 S(t,\lambda)dt\right)\times cos\left(c_2+\int_0^0 T(t,\lambda)dt\right)$$

$$cos\alpha=c_1 cosc_2 \tag{2.49}$$

$\alpha=\frac{\pi}{2},\ \ cos\alpha=0$ ve $cosc_2\neq 0$ ise bu durumda

$$0=c_1 cosc_2,\quad c_1=0$$

olur ki bu aşikar çözümdür.

$\alpha=\frac{\pi}{2},\ \ cos\alpha=0$ ve $c_1\neq 0$ ise bu durumda $c_2$'yi bulalım.

$$0=c_1 cosc_2,\quad cosc_2=0\ \Rightarrow\ c_2=\frac{\pi}{2} \tag{2.50}$$

olur.

$Z'(0,\lambda)=-sin\alpha$ kullanılırsa

$$-sin\alpha = c_1 S(0,\lambda)\exp\left(\int_0^0 S(t,\lambda)dt\right)\times cos\left(c_2+\int_0^0 T(t,\lambda)dt\right)$$

$$-c_1\exp\left(\int_0^0 S(t,\lambda)dt\right)\times sin\left(c_2+\int_0^0 T(t,\lambda)dt\right)T(0,\lambda),$$

$$-sin\alpha = c_1 S(0,\lambda)cosc_2 - c_1 sinc_2 T(0,\lambda)$$

olur. (2.50) kullanılırsa

$$1 = c_1 T(0,\lambda) \;\Rightarrow\; c_1 = \frac{1}{T(0,\lambda)} \tag{2.51}$$

elde edilir. Son olarak (2.50) ve (2.51) değerleri (2.8)'de yerine yazılırsa ve (2.46) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x,\lambda)$ denilirse

$$\varphi(x,\lambda) = \frac{1}{T(0,\lambda)}\exp\left(\int_0^x S(t,\lambda)dt\right)\times cos\left(\frac{\pi}{2}+\int_0^x T(t,\lambda)dt\right),$$

elde edilir ve $cos\left(\frac{\pi}{2}+\beta\right) = -sin\beta$ kullanılırsa

$$\varphi(x,\lambda) = -\frac{1}{T(0,\lambda)}\exp\left(\int_0^x S(t,\lambda)dt\right)\times sin\left(\int_0^x T(t,\lambda)dt\right) \tag{2.52}$$

bulunur.

Notasyonlarla ifade edecek olursak (2.52) eşitliği

$$\varphi(x,\lambda) = -\frac{1}{T(0,\lambda)}E_0(x,\lambda)S_0(x,\lambda)$$

şeklinde düzenlenir ve bulduğumuz (2.30),(2.33) ve (2.35) eşitliklerini $\varphi(x,\lambda)$'da yerine yazarsak

$$\varphi(x,\lambda) = -\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+\,O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

ve gerekli işlemler yapılırsa

$$\varphi(x,\lambda) = -\lambda^{-1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\frac{\lambda^{-1}}{2} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+\frac{\lambda^{-1}}{2}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$\varphi(x,\lambda) = -\lambda^{-1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.47}$$

elde edilir.

**b)** Burada Dirichlet sınır koşulları altında (2.1) − (2.3) özdeğerleri

$$\lambda_n^{1/2} = (n+1) - \frac{1}{(n+1)}A_{2(n+1)} + O(n^{-1}\eta(n)^2) + O\left(n^{-2}\eta(n)\right) \tag{2.53}$$

kullanılacaktır [4]. Böylece

$$\lambda_n^{1/2} = (n+1) + O\left(\eta(n)\right),$$

$$\lambda_n^{-1/2} = \frac{1}{(n+1) + O\left(\eta(n)\right)} = \frac{1}{(n+1)\left[1 + O\left(n^{-1}\eta(n)\right)\right]}$$

$$= \frac{1}{(n+1)} \times \left[1 + O\left(n^{-1}\eta(n)\right)\right] = \frac{1}{(n+1)} + O\left(n^{-2}\eta(n)\right) \tag{2.54}$$

elde edilir. O halde

$$\varphi(x,n) = -\frac{1}{(n+1)} \times sin\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$+O\left(n^{-2}\eta(n)\right) \tag{2.48}$$

bulunur.

**Teorem 2. 8:** $y'' + (\lambda - q)y = 0$ (2.1)

denkleminin

$$y(\pi,\lambda) = cos\beta, \qquad y'(\pi,\lambda) = -sin\beta, \quad \beta = \frac{\pi}{2} \tag{2.55}$$

başlangıç koşullarını sağlayan çözümü

**a)**

$$\psi(x,\lambda) = \lambda^{-1/2} sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.56}$$

**b)**

$$\psi(x,n) = \frac{1}{(n+1)} sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+O\left(n^{-2}\eta(n)\right) \tag{2.57}$$

ile verilir.

**İspat:**

**a)** (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda) = c_1 \mathrm{S(x,\lambda)} \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^x T(t,\lambda)dt\right) T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.55) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.55) ile verilen başlangıç koşullarından

$Z(\pi,\lambda) = cos\beta$ ve $Z'(\pi,\lambda) = -sin\beta$ yazılırsa

$$cos\beta = c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) \tag{2.58}$$

$$\beta = \frac{\pi}{2},\ \ cos\beta = 0,\ \ cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) \neq 0$$

ise (2.58) eşitliği

$$0 = c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) \ \Rightarrow\ c_1 = 0$$

olur ki bu aşikar çözümdür.

$$\beta = \frac{\pi}{2},\ \ cos\beta = 0,\ \ c_1 \neq 0 \ \Rightarrow\ cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) = 0$$

olur buradan

$$c_2 + \int_0^\pi T(t,\lambda)dt = \frac{\pi}{2} \ \Rightarrow\ c_2 = \frac{\pi}{2} - \int_0^\pi T(t,\lambda)dt \tag{2.59}$$

elde edilir.

$$-sin\beta = c_1\, \mathrm{S}(\pi,\lambda)\exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) T(\pi,\lambda)$$

(2.59) kullanılırsa

$$-1 = c_1\, \mathrm{S}(\pi,\lambda)\exp\left(\int_0^\pi S(t,\lambda)dt\right) cos\left(\frac{\pi}{2}\right) - c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) sin\left(\frac{\pi}{2}\right) T(\pi,\lambda)$$

$$1 = c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) T(\pi,\lambda) \ \Rightarrow\ c_1 = \frac{1}{\exp\left(\int_0^\pi S(t,\lambda)dt\right) T(\pi,\lambda)} \tag{2.60}$$

elde edilir. Son olarak (2.59) ve (2.60) değerleri (2.8)'de yerine yazılırsa ve (2.55) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x,\lambda)$ denilirse

$$\psi(x,\lambda)=\frac{\exp\left(\int_0^x S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right)T(\pi,\lambda)}cos\left(\frac{\pi}{2}-\int_0^\pi T(t,\lambda)dt+\int_0^x T(t,\lambda)dt\right)$$

şeklinde bulunur. Sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x,\lambda)=\frac{1}{T(\pi,\lambda)}\exp\left(-\int_x^\pi S(t,\lambda)dt\right)cos\left(\frac{\pi}{2}-\int_x^\pi T(t,\lambda)dt\right)$$

olur ve $cos\left(\frac{\pi}{2}-\beta\right)=sin\beta$ kullanılırsa

$$\psi(x,\lambda)=\frac{1}{T(\pi,\lambda)}\exp\left(-\int_x^\pi S(t,\lambda)dt\right)\times sin\left(\int_x^\pi T(t,\lambda)dt\right) \qquad (2.61)$$

elde edilir.

Notasyonlarla ifade edecek olursak

$$\psi(x,\lambda)=\frac{1}{T(\pi,\lambda)}E_\pi(x,\lambda)S_\pi(x,\lambda)$$

şeklindedir. Bulduğumuz (2.30), (2.36) ve (2.38) değerleri $\psi(x,\lambda)$'da yerine yazılırsa

$$\psi(x,\lambda)=\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

gerekli hesaplamalardan sonra

$$\psi(x,\lambda)=\lambda^{-1/2}sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\frac{\lambda^{-1}}{2}cos\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$\psi(x,\lambda) = \lambda^{-1/2} sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.56)$$

bulunur.

**b)** (2.54)'ten

$$\psi(x,n) = \frac{1}{(n+1)} sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (2.57)$$

elde edilir.

**Teorem 2.9:** Dirichlet sınır şartları altında $\lambda \to \infty$ için

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= \lambda^{-1/2} sin\left(\lambda^{1/2}\pi\right) + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (2.62)$$

ile verilir.

**İspat:**

$\varphi'(x,\lambda)$'yı bulmak için (2.50) ve (2.51)'i (2.9)'da yerine yazalım. Böylece

$$Z'(x,\lambda) = \frac{\mathrm{S(x,\lambda)}}{T(0,\lambda)} \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(\frac{\pi}{2} + \int_0^x T(t,\lambda)dt\right)$$

$$-\frac{1}{T(0,\lambda)} \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(\frac{\pi}{2} + \int_0^x T(t,\lambda)dt\right) T(x,\lambda)$$

olur. Ayrıca

$$cos\left(\frac{\pi}{2} + \int_0^x T(t,\lambda)dt\right) = -\,sin\left(\int_0^x T(t,\lambda)dt\right)$$

ve

$$sin\left(\frac{\pi}{2} + \int_0^x T(t,\lambda)dt\right) = cos\left(\int_0^x T(t,\lambda)dt\right)$$

olduğu kullanılırsa

$$Z'(x,\lambda) = -\frac{1}{T(0,\lambda)}\mathrm{S}(\mathrm{x},\lambda)E_0(x,\lambda)S_0(x,\lambda) - \frac{1}{T(0,\lambda)}E_0(x,\lambda)C_0(x,\lambda)\mathrm{T}(\mathrm{x},\lambda)$$

$$\coloneqq I_1 + I_2$$

şeklinde olur.

$$I_1 = -\frac{1}{T(0,\lambda)}\mathrm{S}(\mathrm{x},\lambda)E_0(x,\lambda)S_0(x,\lambda)$$

$$= -\left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+ O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right] \times \left[\lambda^{-1/2} + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\eta(\lambda)^2\right)\right]$$

gerekli hesaplamalardan sonra

$$= \lambda^{-1/2}sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+\frac{\lambda^{-1}}{2}sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$-\frac{\lambda^{-1}}{2}sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times \int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur.

$$I_2 = -\frac{1}{T(0,\lambda)}E_0(x,\lambda)C_0(x,\lambda)\mathrm{T}(\mathrm{x},\lambda)$$

$$= -\left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}} \int_0^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt\right.$$

$$\left. + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right] \times \left[\lambda^{-1/2} + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$\times \left[cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{1/2} - cos\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right]$$

gerekli hesaplamalardan yapılırsa

$$I_2 = -cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \lambda^{-1/2} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{\lambda^{-1/2}}{2} cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + \frac{\lambda^{-1}}{2} cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$+ \frac{\lambda^{-1/2}}{2} \int_0^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$- \frac{\lambda^{-1}}{2} \int_0^{\pi} sin\left(2\lambda^{1/2}t\right) q(t)dt \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2 = -cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. $I_1$ ve $I_2$'yi $\varphi'(x,\lambda)$'da yerine yazarsak

$$\varphi'(x,\lambda) = -cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.63)$$

şeklinde olur.

Şimdi $\psi'(x,\lambda)$'yı bulmak için (2.59) ve (2.60)'ı (2.9)'da yerine yazalım.

$$Z'(x,\lambda)=\frac{\mathrm{S}(\mathrm{x},\lambda)\exp\left(\int_0^x S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right)T(\pi,\lambda)}\times cos\left(\frac{\pi}{2}-\int_0^\pi T(t,\lambda)dt+\int_0^x T(t,\lambda)dt\right)$$

$$-\frac{\exp\left(\int_0^x S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right)T(\pi,\lambda)}$$

$$\times\, sin\left(\frac{\pi}{2}-\int_0^\pi T(t,\lambda)dt+\int_0^x T(t,\lambda)dt\right)T(x,\lambda)$$

sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$Z'(x,\lambda)=\frac{\mathrm{S}(\mathrm{x},\lambda)}{T(\pi,\lambda)}\exp\left(-\int_x^\pi S(t,\lambda)dt\right)\times cos\left(\frac{\pi}{2}-\int_x^\pi T(t,\lambda)dt\right)$$

$$-\frac{1}{T(\pi,\lambda)}\exp\left(-\int_x^\pi S(t,\lambda)dt\right)\times sin\left(\frac{\pi}{2}-\int_x^\pi T(t,\lambda)dt\right)T(x,\lambda),$$

olur.

$$cos\left(\frac{\pi}{2}-\int_x^\pi T(t,\lambda)dt\right)=sin\left(\int_x^\pi T(t,\lambda)dt\right),$$

ve

$$sin\left(\frac{\pi}{2}-\int_x^\pi T(t,\lambda)dt\right)=cos\left(\int_x^\pi T(t,\lambda)dt\right),$$

olduğu kullanılırsa

$$Z'(x,\lambda)=\frac{1}{T(\pi,\lambda)}\mathrm{S}(\mathrm{x},\lambda)\exp\left(-\int_x^\pi S(t,\lambda)dt\right)sin\left(\int_x^\pi T(t,\lambda)dt\right)$$

$$-\frac{1}{T(\pi,\lambda)}\exp\left(-\int_x^\pi S(t,\lambda)dt\right)cos\left(\int_x^\pi T(t,\lambda)dt\right)$$

elde edilir ve notasyonlarla ifade edersek

$$Z'(x,\lambda)=\frac{1}{T(\pi,\lambda)}\mathrm{S(x,\lambda)}E_{\pi}(x,\lambda)S_{\pi}(x,\lambda)-\frac{1}{T(\pi,\lambda)}E_{\pi}(x,\lambda)C_{\pi}(x,\lambda)\mathrm{T(x,\lambda)}$$

$$\coloneqq I_1-I_2$$

şeklinde olur.

$$I_1=\frac{1}{T(\pi,\lambda)}\mathrm{S(x,\lambda)}E_{\pi}(x,\lambda)S_{\pi}(x,\lambda)$$

$$=\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]\times\left[\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=-\lambda^{-1/2}sin\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)$$

$$-\frac{\lambda^{-1}}{2}sin\left(2\lambda^{1/2}x+\xi_x\right)\times sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)$$

$$\times cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1=O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanır.

$$I_2=-\frac{1}{T(\pi,\lambda)}\exp\left(-\int_x^{\pi}S(t,\lambda)dt\right)cos\left(\int_x^{\pi}T(t,\lambda)dt\right)$$

$$=-\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[\lambda^{-1/2}+O\left(\lambda^{-1}\eta(\lambda)\right)\right]\times\left[\lambda^{1/2}+cos\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]$$

gerekli hesaplamalardan sonra

$$= -cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) + \frac{\lambda^{-1/2}}{2}\ cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times\ cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) + \frac{\lambda^{-1/2}}{2} cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times\ cos\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2 = -cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

bulunur. $I_1$ ve $I_2$'yi $\psi'(x,\lambda)$'da yerine yazarsak

$$\psi'(x,\lambda) = -cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.64)$$

elde edilir.

Bulduğumuz (2.47), (2.56), (2.63) ve (2.64) eşitlikleri $w_x(\varphi,\psi)$'da yerine yazılırsa

$$w_x(\varphi,\psi) =$$

$$= \begin{bmatrix} \lambda^{-1/2} \\ \times\ sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^{x} q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

$$+ \begin{bmatrix} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^{x} q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} \lambda^{-1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

şeklinde olur. Böylece

$$w(\lambda) = \lambda^{-1/2}sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^{x} q(t)dt\right)$$

$$\times\ cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right)$$

$$+\lambda^{-1/2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\big(\lambda^{-1}\eta(\lambda)\big)$$

$$=\lambda^{-1/2}\left[sin\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)\right]+O\big(\lambda^{-1}\eta(\lambda)\big)$$

bulunur. $\int_0^\pi q(t)dt=0\ (ortalama\ değer)$ olduğundan

$$w(\lambda)=\lambda^{-1/2}sin\big(\lambda^{1/2}\pi\big)+O\big(\lambda^{-1}\eta(\lambda)\big) \tag{2.62}$$

elde edilir.

**Teorem 2.10:**

**i)** Dirichlet sınır şartları altında $\lambda\to\infty$ ve $0\le x\le y\le\pi$ için

$$G(x,y,\lambda)=$$

$$=\frac{\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right]\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right]}{\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.65),$$

**ii)**

$$G(x,y,n)=\frac{1}{(n+1)}sin\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$\times sin\left((n+1)(\pi-y)-\frac{1}{2(n+1)}\int_y^\pi q(t)dt\right)$$

$$+O\big(n^{-3}\eta(n)\big) \tag{2.66}$$

ile verilir.

**İspat:**

**i)**

$$G(x,y,\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}+O\left(\lambda^{-k}\eta(\lambda)\right), & 0\le x\le y\le\pi\\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}+O\left(\lambda^{-k}\eta(\lambda)\right), & 0\le y\le x\le\pi\end{cases}$$

şeklinde idi. Bulduğumuz (2.62), (2.63) ve (2.64)'ü $G(x,y,\lambda)'da$ yerine yazarsak

$0\le x\le y\le\pi$ için

$$G(x,y,\lambda)=\frac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}$$

$$=\frac{\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\right]\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\right]}{\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.65)$$

bulunur. Benzer şekilde $0\le y\le x\le\pi$ için de

$$G(x,y,\lambda)=\frac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}$$

elde edilir.

**ii)** Bulduğumuz $G(x,y,\lambda)$ eşitliğinde reversion uygularsak

$$G(x,y,\lambda)=$$

$$=\frac{\left[\begin{array}{c}\lambda^{-1/2}sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\\ \times\, sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{array}\right]}{sin(\lambda^{1/2}\pi)+O\left(\lambda^{-1}\eta(\lambda)\right)}$$

$$=\frac{\left[\begin{array}{c}\lambda^{-1/2}sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\\ \times\, sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{array}\right]}{sin(\lambda^{1/2}\pi)\times\left[1+O\left(\lambda^{-1}\eta(\lambda)\right)\right]}$$

$$= \frac{\begin{bmatrix} \lambda^{-1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ \times\, sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}}{sin(\lambda^{1/2}\pi)} \times \left[1 + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$= \frac{\begin{bmatrix} \lambda^{-1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ \times\, sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \end{bmatrix}}{sin(\lambda^{1/2}\pi)} + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. (2.54)’ten

$$G(x,y,n) =$$

$$\frac{\begin{bmatrix} \frac{1}{(n+1) + o(\eta(n))} \\ \times\, sin\left((n+1)x - \frac{1}{2\left((n+1) + o(\eta(n))\right)}\int_0^x q(t)dt\right) \end{bmatrix} \times \left[sin\left((n+1)(\pi - y) - \frac{1}{2\left((n+1) + o(\eta(n))\right)}\int_y^\pi q(t)dt\right)\right] + O\left(n^{-3}\eta(n)\right)}{sin\left((n+1)\pi + O(\eta(n))\right)}$$

dir, burada $sin\left((n+1)\pi + O(\eta(n))\right)$ değerini hesaplayalım.

$$sin\left((n+1)\pi + O(\eta(n))\right) =$$

$$= cos((n+1)\pi) sin\left(O(\eta(n))\right)$$

$$+ sin((n+1)\pi) cos\left(O(\eta(n))\right) = O(\eta(n))$$

Bulduğumuz değeri $G(x,y,n)$’de yerine yazarsak

$$G(x,y,n)=\frac{1}{(n+1)}sin\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$\times sin\left((n+1)(\pi-y)-\frac{1}{2(n+1)}\int_y^\pi q(t)dt\right)$$

$$+O\left(n^{-3}\eta(n)\right) \qquad (2.66)$$

olarak bulunur.

Aşağıdaki teoremlerde Neumann koşulları altında Green fonksiyonu elde edilecektir.

**Teorem 2.11:** $y''+(\lambda-q)y=0$ (2.1)

denkleminin

$$y(0,\lambda)=cos\alpha, \qquad y'(0,\lambda)=-sin\alpha, \quad \alpha=0 \qquad (2.67)$$

başlangıç koşullarını sağlayan çözümü

**i)**

$$\varphi(x,\lambda)=cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.68)$$

**ii)**

$$\varphi(x,n)=cos\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$-sin\left((n+1)x-\frac{1}{2(n+1)}\int_0^x q(t)dt\right)+O\left(n^{-2}\eta(n)\right) \qquad (2.69)$$

şeklindedir.

**İspat:**

**i)** (1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda) = c_1\, \mathrm{S(x,}\lambda\mathrm{)exp}\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^x T(t,\lambda)dt\right) T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.67) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.67) ile verilen başlangıç koşullarından

$$Z(0,\lambda) = cos\alpha \;\Rightarrow cos\alpha = c_1 \exp\left(\int_0^0 S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^0 T(t,\lambda)dt\right)$$

$$cos\alpha = c_1 cosc_2 \;\Rightarrow\; 1 = c_1 cosc_2$$

$$\Rightarrow c_1 = \frac{1}{cosc_2} \tag{2.70}$$

$Z'(0,\lambda) = -sin\alpha$ kullanılırsa

$$-sin\alpha = c_1 S(0,\lambda) \exp\left(\int_0^0 S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^0 T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^0 S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^0 T(t,\lambda)dt\right) T(0,\lambda),$$

$$-sin\alpha = c_1 S(0,\lambda) cosc_2 - c_1 sinc_2 T(0,\lambda),$$

$$0 = \frac{1}{cosc_2} S(0,\lambda) cosc_2 - \frac{1}{cosc_2} sinc_2 T(0,\lambda),$$

$$S(0,\lambda) = tanc_2 T(0,\lambda),$$

$$c_2 = tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] \tag{2.71}$$

olur. (2.71) eşitliği (2.70)'de yerine yazılırsa

$$c_1 = \frac{1}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \tag{2.72}$$

bulunur. Son olarak (2.71) ve (2.72) değerleri $Z(x,\lambda)$'da yerine yazılırsa ve (2.67) ile verilen başlangıç koşullarını sağlayan çözüme $\varphi(x,\lambda)$ denilirse

$$\varphi(x,\lambda) = \frac{1}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \exp\left(\int_0^x S(t,\lambda)dt\right)$$

$$\times\, cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] + \int_0^x T(t,\lambda)dt\right)$$

$$= \frac{exp\left(\int_0^x S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)}\left[cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)cos\left(\int_0^x T(t,\lambda)dt\right)\right.$$
$$\left. - sin\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)sin\left(\int_0^x T(t,\lambda)dt\right)\right]$$

şeklinde elde edilir ve düzenlemeler yapılırsa

$$\varphi(x,\lambda) = exp\left(\int_0^x S(t,\lambda)dt\right)cos\left(\int_0^x T(t,\lambda)dt\right)$$
$$-exp\left(\int_0^x S(t,\lambda)dt\right)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]sin\left(\int_0^x T(t,\lambda)dt\right) \tag{2.73}$$

olarak bulunur.

Natosyonlarla ifade edecek olursak

$$\varphi(x,\lambda) = E_0(x,\lambda)C_0(x,\lambda) - E_0(x,\lambda)S_0(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]$$

$$:= I_1 - I_2$$

şeklinde olur.

$$I_1 = E_0(x,\lambda)C_0(x,\lambda)$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1=cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

ve

$$I_2=E_0(x,\lambda)S_0(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]$$

$$=\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt\right.$$

$$\left.+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2=sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. $I_1$ ve $I_2$ eşitliklerini $\varphi(x,\lambda)$'da yerine yazarsak

$$\varphi(x,\lambda) = cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.68}$$

bulunur.

**ii)** (2.54)'ten

$$\varphi(x,n) = cos\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$-sin\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right) + O\big(n^{-2}\eta(n)\big) \tag{2.66}$$

elde edilir.

**Teorem 2.12:** $y'' + (\lambda - q)y = 0$ (2.1)

denkleminin

$$y(\pi,\lambda) = cos\beta, \qquad y'(\pi,\lambda) = -sin\beta, \quad \beta = 0 \tag{2.74}$$

başlangıç koşullarını sağlayan çözümü

**i)**

$$\psi(x,\lambda) = cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.75),$$

**ii)**

$$\psi(x,n) = cos\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right) + O\big(n^{-2}\eta(n)\big) \tag{2.76}$$

ile verilir.

**İspat:**

**i)** (2.1) denkleminin sıfırdan farklı reel değerli bir çözümü

$$Z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right) \tag{2.8}$$

ve

$$Z'(x,\lambda) = c_1 \, \mathrm{S}(\mathrm{x},\lambda) \exp\left(\int_0^x S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^x T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^x T(t,\lambda)dt\right) T(x,\lambda) \tag{2.9}$$

şeklindedir [4]. (2.74) ile verilen başlangıç koşulları sağlanacak şekilde $c_1$ ve $c_2$ sabitlerini belirleyelim. (2.74) ile verilen başlangıç koşullarından

$$Z(\pi,\lambda) = cos\beta$$

$$1 = c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) \tag{2.77}$$

$Z'(\pi,\lambda) = -sin\beta$ kullanılırsa

$$0 = c_1 S(\pi,\lambda) \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)$$

$$-c_1 \exp\left(\int_0^\pi S(t,\lambda)dt\right) \times sin\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) T(\pi,\lambda) \tag{2.78}$$

(2.77) eşitliğini (2.78)'de yerine yazarsak

$$0 = \frac{S(\pi,\lambda)\exp\left(\int_0^\pi S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)} \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)$$

$$-\frac{\exp\left(\int_0^\pi S(t,\lambda)dt\right)}{\exp\left(\int_0^\pi S(t,\lambda)dt\right) \times cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)} \times sin\left(c_2 + \int_0^\pi T(t,\lambda)dt\right) T(\pi,\lambda)$$

$$= S(\pi,\lambda) - \frac{sin\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)}{cos\left(c_2 + \int_0^\pi T(t,\lambda)dt\right)} T(\pi,\lambda),$$

$$\frac{S(\pi,\lambda)}{T(\pi,\lambda)} = tan\left(c_2 + \int_0^{\pi} T(t,\lambda)dt\right)$$

bulunur ve

$$c_2 = tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_0^{\pi} T(t,\lambda)dt \tag{2.79}$$

olarak elde edilir. (2.79) eşitliğini (2.77)'de yerine yazarsak

$$1 = c_1 \exp\left(\int_0^{\pi} S(t,\lambda)dt\right) \times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_0^{\pi} T(t,\lambda)dt + \int_0^{\pi} T(t,\lambda)dt\right)$$

$$c_1 = \frac{1}{\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)} \tag{2.80}$$

bulunur. Son olarak (2.79) ve (2.80) değerleri $Z(x,\lambda)$'da yerine yazılırsa ve (2.74) ile verilen başlangıç koşullarını sağlayan çözüme $\psi(x,\lambda)$ denilirse

$$\psi(x,\lambda) = \frac{\exp\left(\int_0^{x} S(t,\lambda)dt\right)}{\exp\left(\int_0^{\pi} S(t,\lambda)dt\right) cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)}$$

$$\times\, cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_0^{\pi} T(t,\lambda)dt + \int_0^{x} T(t,\lambda)dt\right)$$

olur, sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$\psi(x,\lambda) = \frac{\exp\left(-\int_x^{\pi} S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)} \times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_x^{\pi} T(t,\lambda)dt\right)$$

$$= \frac{\exp\left(-\int_x^{\pi} S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)} \times \left[cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right) cos\left(\int_x^{\pi} T(t,\lambda)dt\right)\right.$$

$$\left. + sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right) sin\left(\int_x^{\pi} T(t,\lambda)dt\right)\right]$$

$$\psi(x,\lambda) = exp\left(-\int_x^{\pi} S(t,\lambda)dt\right)$$

$$\times\left[cos\left(\int_x^{\pi}T(t,\lambda)dt\right)+\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]sin\left(\int_x^{\pi}T(t,\lambda)dt\right)\right] \tag{2.81}$$

elde ederiz.

Natosyonlarla ifade edecek olursak

$$\psi(x,\lambda)=E_{\pi}(x,\lambda)C_{\pi}(x,\lambda)+E_{\pi}(x,\lambda)S_{\pi}(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]$$

$$:=I_1-I_2$$

şeklinde olur. Buradan

$$I_1=E_{\pi}(x,\lambda)C_{\pi}(x,\lambda)$$

$$=\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$=cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-3/2}\eta(\lambda)\right)$$

$$I_1=cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklindedir ve

$$I_2=E_{\pi}(x,\lambda)S_{\pi}(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]$$

$$=\left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^{\pi}q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$= sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-3/2}\eta(\lambda)\right)$$

$$I_2 = sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde elde edilir. $I_1$ ve $I_2$ eşitlikleri $\psi(x,\lambda)$'da yerine yazılırsa

$$\psi(x,\lambda) = cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.75}$$

bulunur.

**ii)** (2.54)'ten

$$\psi(x,n) = cos\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+sin\left((n+1)(\pi - x) - \frac{1}{2(n+1)}\int_x^\pi q(t)dt\right) + O\left(n^{-2}\eta(n)\right) \tag{2.76}$$

elde edilir.

**Teorem 2.13:** Neumann sınır şartları altında $\lambda \to \infty$ için

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= 2\lambda^{1/2}sin\left(\lambda^{1/2}\pi\right) + O\left(\lambda^{-1}\eta(\lambda)\right) \tag{2.82}$$

ile verilir.

**İspat:** $\varphi'(x,\lambda)$'yı bulmak için (2.71) ve (2.72)'yi $Z'(x,\lambda)$'da yerine yazalım.

$$Z'(x,\lambda) = \frac{S(x,\lambda)exp\left(\int_0^x S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)} \times cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right] + \int_0^x T(t,\lambda)dt\right)$$

$$-\frac{exp\left(\int_0^x S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)}\times sin\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]+\int_0^x T(t,\lambda)dt\right)T(x,\lambda)$$

$$=S(x,\lambda)E_0(x,\lambda)\left[\frac{\begin{array}{c}cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)\times cos\left(\int_0^x T(t,\lambda)dt\right)\\ -sin\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)\times sin\left(\int_0^x T(t,\lambda)dt\right)\end{array}}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)}\right]$$

$$-E_0(x,\lambda)\left[\frac{\begin{array}{c}sin\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)\times cos\left(\int_0^x T(t,\lambda)dt\right)\\ +cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)\times sin\left(\int_0^x T(t,\lambda)dt\right)\end{array}}{cos\left(tan^{-1}\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]\right)}\right]T(x,\lambda)$$

şeklinde olur ve notasyonlar yardımıyla

$$=S(x,\lambda)E_0(x,\lambda)\left[cos\left(\int_0^x T(t,\lambda)dt\right)-\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]sin\left(\int_0^x T(t,\lambda)dt\right)\right]$$

$$-E_0(x,\lambda)\left[\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]cos\left(\int_0^x T(t,\lambda)dt\right)+sin\left(\int_0^x T(t,\lambda)dt\right)\right]T(x,\lambda)$$

$$Z'(x,\lambda)=S(x,\lambda)E_0(x,\lambda)C_0(x,\lambda)-S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]$$

$$-E_0(x,\lambda)C_0(x,\lambda)T(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]-E_0(x,\lambda)S_0(x,\lambda)T(x,\lambda)$$

gösterilir.

$$Z'(x,\lambda)\coloneqq I_1-I_2-I_3-I_4$$

olarak tanımlarsak

$$I_1=S(x,\lambda)E_0(x,\lambda)C_0(x,\lambda)$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt \right.$$

$$\left. + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right] \times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right]$$

$$\times \left[cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$= -sin\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$-\frac{1}{2\lambda^{1/2}} sin\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) + \frac{1}{2\lambda^{1/2}} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt$$

$$\times sin\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

olarak bulunur. İkinci olarak

$$I_2 = S(x,\lambda)E_0(x,\lambda)S_0(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}} \int_0^\pi sin\left(2\lambda^{1/2}t\right) q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$= -sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}} \int_0^x q(t)dt\right)$$

$$-\frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\big(2\lambda^{1/2}t\big)q(t)dt\times$$

$$\times sin\big(2\lambda^{1/2}x+\xi_x\big)\times sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2=\ O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. Üçüncü olarak

$$I_3=E_0(x,\lambda)C_0(x,\lambda)T(x,\lambda)\left[\frac{S(0,\lambda)}{T(0,\lambda)}\right]$$

$$=\left[1+\frac{1}{2\lambda^{1/2}}cos\big(2\lambda^{1/2}x+\xi_x\big)-\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\big(2\lambda^{1/2}t\big)q(t)dt+O\big(\lambda^{-1/2}\eta(\lambda)^2\big)\right]$$

$$\times\left[cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\big(\lambda^{-1/2}\eta(\lambda)^2\big)\right]\times\left[\ O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times\left[\lambda^{1/2}-cos\big(2\lambda^{1/2}x+\xi_x\big)+O\big(\eta(\lambda)^2\big)\right]$$

$$=\lambda^{1/2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+\frac{1}{2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times cos\big(2\lambda^{1/2}x+\xi_x\big)-\frac{1}{2}\int_0^\pi sin\big(2\lambda^{1/2}t\big)q(t)dt$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-cos\big(2\lambda^{1/2}x+\xi_x\big)$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-\frac{1}{2\lambda^{1/2}}cos\big(2\lambda^{1/2}x+\xi_x\big)$$

$$\times cos\big(2\lambda^{1/2}x+\xi_x\big)\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+\frac{1}{2\lambda^{1/2}}\int_0^\pi sin\big(2\lambda^{1/2}t\big)q(t)dt\times cos\big(2\lambda^{1/2}x+\xi_x\big)$$

$$\times cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3 = \lambda^{1/2} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

hesaplanır ve son olarak

$$I_4 = E_0(x,\lambda)S_0(x,\lambda)T(x,\lambda)$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[\lambda^{1/2} - cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\eta(\lambda)^2\right)\right]$$

$$= \lambda^{1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + \frac{1}{2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) - \frac{1}{2}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) - \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+ \frac{1}{2\lambda^{1/2}}\int_0^\pi sin\left(2\lambda^{1/2}t\right)q(t)dt \times cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_4 = \lambda^{1/2} sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir. $I_1$, $I_2$, $I_3$ ve $I_4$ değerleri $\varphi'(x,\lambda)$'da yerine yazılırsa

$$\varphi'(x,\lambda) = -\lambda^{1/2}cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\lambda^{1/2}sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-3/2}\eta(\lambda)\right) \qquad (2.83)$$

bulunur.

$\psi'(x,\lambda)$'yı elde etmek için (2.79) ve (2.80)'i $Z'(x,\lambda)$'da yerine yazalım.

$$Z'(x,\lambda) = \frac{S(x,\lambda)exp\left(\int_0^x S(t,\lambda)dt\right)}{exp\left(\int_0^\pi S(t,\lambda)dt\right)cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)}$$

$$\times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_0^\pi T(t,\lambda)dt + \int_0^x T(t,\lambda)dt\right)$$

$$-\frac{exp\left(\int_0^x S(t,\lambda)dt\right)}{exp\left(\int_0^\pi S(t,\lambda)dt\right)cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)}$$

$$\times sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_0^\pi T(t,\lambda)dt + \int_0^x T(t,\lambda)dt\right)T(x,\lambda)$$

olur ve sınırlar üzerinde gerekli düzenlemeler yapılırsa

$$Z'(x,\lambda) = \frac{S(x,\lambda)exp\left(-\int_x^\pi S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)} \times cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_x^\pi T(t,\lambda)dt\right)$$

$$-\frac{exp\left(-\int_x^\pi S(t,\lambda)dt\right)}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)} \times sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right] - \int_x^\pi T(t,\lambda)dt\right)T(x,\lambda)$$

$$= S(x,\lambda)exp\left(-\int_x^\pi S(t,\lambda)dt\right)\left[\frac{\begin{array}{c} cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)cos\left(\int_x^\pi T(t,\lambda)dt\right) \\ +sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)sin\left(\int_x^\pi T(t,\lambda)dt\right) \end{array}}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)}\right]$$

$$-exp\left(-\int_x^\pi S(t,\lambda)dt\right)\left[\frac{\begin{array}{c} sin\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)cos\left(\int_x^\pi T(t,\lambda)dt\right) \\ -cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)sin\left(\int_x^\pi T(t,\lambda)dt\right)\end{array}}{cos\left(tan^{-1}\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]\right)}\right]T(x,\lambda)$$

$$= S(x,\lambda)exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$\times\left[cos\left(\int_x^\pi T(t,\lambda)dt\right)+\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]sin\left(\int_x^\pi T(t,\lambda)dt\right)\right]$$

$$-exp\left(-\int_x^\pi S(t,\lambda)dt\right)$$

$$\times\left[\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]cos\left(\int_x^\pi T(t,\lambda)dt\right)-sin\left(\int_x^\pi T(t,\lambda)dt\right)\right]T(x,\lambda)$$

elde edilir. Notasyonlarla ifade edecek olursak

$$Z'(x,\lambda) = S(x,\lambda)E_\pi(x,\lambda)C_\pi(x,\lambda)+S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]$$

$$-E_\pi(x,\lambda)C_\pi(x,\lambda)T(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]+E_\pi(x,\lambda)S_\pi(x,\lambda)T(x,\lambda)$$

şeklinde buluruz.

$$Z'(x,\lambda) \coloneqq I_1 - I_2 - I_3 - I_4$$

olarak tanımlarsak, $I_1$ değerimiz

$$I_1 = S(x,\lambda)E_\pi(x,\lambda)C_\pi(x,\lambda)$$

$$= \left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[-sin\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]$$

$$\times \left[ cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \right]$$

$$= -sin\left(2\lambda^{1/2}x + \xi_x\right) \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)$$

$$- \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_1 = O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanır. İkinci olarak $I_2$ değerimiz

$$I_2 = S(x,\lambda)E_\pi(x,\lambda)S_\pi(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]$$

$$= \left[1 + \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times \left[-sin\left(2\lambda^{1/2}x + \xi_x\right) + O(\eta(\lambda)^2)\right] \times \left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times \left[ sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)^2\right) \right]$$

$$= -sin\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right)$$

$$- \frac{1}{2\lambda^{1/2}} cos\left(2\lambda^{1/2}x + \xi_x\right) \times sin\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}} \int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_2 = O\left(\lambda^{-1/2}\eta(\lambda)\right),$$

üçüncü olarak $I_3$ değerimiz

$$I_3 = E_\pi(x,\lambda)C_\pi(x,\lambda)T(x,\lambda)\left[\frac{S(\pi,\lambda)}{T(\pi,\lambda)}\right]$$

$$= \left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]\times\left[O\left(\lambda^{-1/2}\eta(\lambda)\right)\right]$$

$$\times\left[cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[\lambda^{1/2}-cos\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]$$

$$= \lambda^{1/2}cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+\frac{1}{2}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)-cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)-\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)\times cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_3 = \lambda^{1/2}cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

ve son olarak $I_4$ eşitliğimiz

$$I_4 = E_\pi(x,\lambda)S_\pi(x,\lambda)T(x,\lambda)$$

$$= \left[1+\frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x+\xi_x\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)^2\right)\right]$$

$$\times\left[\lambda^{1/2}-cos\left(2\lambda^{1/2}x+\xi_x\right)+O(\eta(\lambda)^2)\right]$$

$$= \lambda^{1/2}sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+\frac{1}{2}cos\left(2\lambda^{1/2}x+\xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) - \frac{1}{2\lambda^{1/2}}cos\left(2\lambda^{1/2}x + \xi_x\right)$$

$$\times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \times cos\left(2\lambda^{1/2}x + \xi_x\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

$$I_4 = \lambda^{1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

şeklinde hesaplanmış olur. $I_1$, $I_2$, $I_3$ ve $I_4$ değerleri $\psi'(x,\lambda)$'da yerine yazılırsa

$$\psi'(x,\lambda) = -\lambda^{1/2}cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)$$

$$+\lambda^{1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) + O\left(\lambda^{-3/2}\eta(\lambda)\right) \qquad (2.84)$$

elde edilir.

Bulduğumuz (2.68),(2.75),(2.80) ve (2.84) eşitlikleri $w_x(\varphi,\psi)$'da yerine yazılırsa

$$w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= \begin{bmatrix} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} -\lambda^{1/2}cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +\lambda^{1/2}sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

$$- \begin{bmatrix} -\lambda^{1/2}cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ -\lambda^{1/2}sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix} \times \begin{bmatrix} cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{bmatrix}$$

gerekli hesaplamalardan sonra

$$=\begin{bmatrix} -\lambda^{1/2}cos\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\times cos\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\ (1a) \\ +\lambda^{1/2}sin\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\times cos\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\ (1b) \\ +\lambda^{1/2}cos\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\times sin\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\ (1b) \\ -\lambda^{1/2}sin\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\times sin\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\ (1a) \end{bmatrix}$$

$$+\begin{bmatrix} +\lambda^{1/2}cos\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\times cos\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\ (1c) \\ +\lambda^{1/2}cos\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\times sin\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\ (1d) \\ +\lambda^{1/2}sin\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\times cos\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\ (1d) \\ +\lambda^{1/2}sin\left(\lambda^{1/2}x-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_0^x q(t)dt\right)\times sin\left(\lambda^{1/2}(\pi-x)-\dfrac{1}{2\lambda^{1/2}}\displaystyle\int_x^\pi q(t)dt\right)\ (1c) \end{bmatrix}$$

$$+O\left(\lambda^{-1}\eta(\lambda)\right)$$

elde edilir ve kosinüs ve sinüs toplam fark formülleri kullanırsa $w(\lambda)$ eşitliği

$$w(\lambda)=-\lambda^{1/2}cos\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt+\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \quad (1a)$$

$$+\lambda^{1/2}sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt+\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \quad (1b)$$

$$+\lambda^{1/2}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt+\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \quad (1c)$$

$$+\lambda^{1/2}sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt+\lambda^{1/2}(\pi-x)\right.$$
$$\left.-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \tag{1d}$$
$$+O\left(\lambda^{-1}\eta(\lambda)\right)$$

ve gerekli düzenlemeler yapılırsa $w(\lambda)$ eşitliğimiz

$$w(\lambda)=-\lambda^{1/2}cos\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$
$$+\lambda^{1/2}sin\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)+\lambda^{1/2}cos\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)$$
$$+\lambda^{1/2}sin\left(\lambda^{1/2}\pi-\frac{1}{2\lambda^{1/2}}\int_0^\pi q(t)dt\right)+O\left(\lambda^{-1}\eta(\lambda)\right)$$
$$w(\lambda)=2\lambda^{1/2}sin\left(\lambda^{1/2}\pi\right)+O\left(\lambda^{-1}\eta(\lambda)\right) \tag{2.82}$$

elde edilir.

**Teorem 2.14:**

**i)** Neumann sınır şartları altında $\lambda\to\infty$ ve $0\le x\le y\le\pi$ için

$$G(x,y,\lambda)=$$
$$=\frac{\begin{pmatrix} cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\\ -sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\end{pmatrix}\times\begin{pmatrix} cos\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)\\ +sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)\end{pmatrix}}{2\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$
$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.85}$$

**ii)**

$$G(x,y,n)=$$

$$=cos\left(\begin{array}{c}(n+1)\pi+(n+1)(x-y)\\ -\frac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt+\int_y^\pi q(t)dt\right)\end{array}\right)$$

$$+sin\left(\begin{array}{c}(n+1)(\pi-y)-(n+1)x\\ +\frac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt-\int_y^\pi q(t)dt\right)\end{array}\right)$$

$$+O\big(n^{-3}\eta(n)\big) \quad (2.86)$$

ile verilir.

**İspat:**

**i)**

$$G(x,y,\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}+O\Big(\lambda^{-k}\eta(\lambda)\Big), & 0\le x\le y\le\pi\\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}+O\Big(\lambda^{-k}\eta(\lambda)\Big), & 0\le y\le x\le\pi\end{cases}$$

şeklinde idi. Bulduğumuz (2.82), (2.83) ve (2.84)'ü Green fonksiyonunda yerine yazarsak

$0\le x\le y\le\pi$ için

$$G(x,y,\lambda)=\frac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}$$

$$=\frac{\begin{pmatrix}cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\\ -sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)\end{pmatrix}\times\begin{pmatrix}cos\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)\\ +sin\left(\lambda^{1/2}(\pi-y)-\frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right)\end{pmatrix}}{2\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.85)$$

buluruz. Benzer şekilde $0\le y\le x\le\pi$ içinde

$$G(x,y,\lambda) = \frac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}$$

elde edilir.

**ii)** Bulduğumuz $G(x,y,\lambda)$'ya reversion uygularsak

$$G(x,y,\lambda) =$$

$$= \frac{\left[\begin{array}{c} cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times \ sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ -sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times \ cos\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ -sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) \times \ sin\left(\lambda^{1/2}(\pi - y) - \frac{1}{2\lambda^{1/2}}\int_y^\pi q(t)dt\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{array}\right]}{2\lambda^{1/2}sin(\lambda^{1/2}\pi) \times \left[1 + O\left(\lambda^{-1}\eta(\lambda)\right)\right]}$$

$$= \frac{\left[\begin{array}{c} cos\left(\lambda^{1/2}\pi + \lambda^{1/2}(x - y) - \frac{1}{2\lambda^{1/2}}\left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right)\right) \\ +sin\left(\lambda^{1/2}(\pi - y) - \lambda^{1/2}x + \frac{1}{2\lambda^{1/2}}\left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right)\right) \\ +O\left(\lambda^{-1/2}\eta(\lambda)\right) \end{array}\right]}{2\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$

$$\times \left[1 + O\left(\lambda^{-1}\eta(\lambda)\right)\right]$$

$$= \frac{\left[\begin{array}{c} cos\left(\lambda^{1/2}\pi + \lambda^{1/2}(x - y) - \frac{1}{2\lambda^{1/2}}\left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right)\right) \\ +sin\left(\lambda^{1/2}(\pi - y) - \lambda^{1/2}x + \frac{1}{2\lambda^{1/2}}\left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right)\right) \end{array}\right]}{2\lambda^{1/2}sin(\lambda^{1/2}\pi)}$$

$$\times O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

elde edilir ve (2.54)'ten

$$G(x,y,n) =$$

$$= \frac{\begin{bmatrix} cos\begin{pmatrix} (n+1)\pi + (n+1)(x-y) \\ -\dfrac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right) \end{pmatrix} \\ +sin\begin{pmatrix} (n+1)(\pi-y)-(n+1)x \\ +\dfrac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right) \end{pmatrix} \\ +O(n^{-3}\eta(n)) \end{bmatrix}}{2(n+1)\times sin\big((n+1)\pi\big)}$$

bulunur. Burada $sin\left((n+1)\pi + O\big(\eta(n)\big)\right)$ değerini hesaplayıp $G(x,y,n)$ ‘de yerine yazalım.

$$sin\left((n+1)\pi + O\big(\eta(n)\big)\right) =$$

$$= cos\big((n+1)\pi\big)sin\left(O\big(\eta(n)\big)\right)$$

$$+sin\big((n+1)\pi\big)cos\left(o\big(\eta(n)\big)\right)$$

$$= O\big(\eta(n)\big)$$

olarak elde edilir.

$$G(x,y,n) =$$

$$= cos\begin{pmatrix} (n+1)\pi + (n+1)(x-y) \\ -\dfrac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt + \int_y^\pi q(t)dt\right) \end{pmatrix}$$

$$+sin\begin{pmatrix} (n+1)(\pi-y)-(n+1)x \\ +\dfrac{1}{2\big((n+1)+O(\eta(n))\big)}\left(\int_0^x q(t)dt - \int_y^\pi q(t)dt\right) \end{pmatrix}$$

$$+O\big(n^{-3}\eta(n)\big) \qquad (2.86)$$

bulunur.

# 3. POTANSİYEL FONKSİYONUN q(x)= $x^{-1/2} - \frac{2}{\sqrt{\pi}}$ OLMASI DURUMU

**Teorem 3.1:** $\lambda \to \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right) y = 0 \tag{3.1}$$

denkleminin

$$y(0,\lambda) = cos\alpha, \quad y'(0,\lambda) = -sin\alpha \qquad \alpha \in [0,\pi] \tag{3.2}$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned}\varphi(x,\lambda) = cos\alpha \times cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)\\ -cos\alpha \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)\\ -\lambda^{-1/2} sin\alpha \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned} \tag{3.3}$$

ile verilir.

**İspat:** Bir önceki bölümde,

$$\int_0^{\pi} q(t)dt = 0$$

olmak koşuluyla

$$y'' + (\lambda - q)y = 0, \quad x \in [0,\pi]$$

$$y(0,\lambda) = cos\alpha, \quad y'(0,\lambda) = -sin\alpha, \qquad \alpha \in [0,\pi]$$

probleminin çözümünü

$$\varphi(x,\lambda) = cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$-\lambda^{-1/2}sin\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (2.39)$$

olarak bulmuştuk. Öncelikle

$$\int_0^\pi \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)dx = 0$$

sağlanır. $\int_0^x q(t)dt$ değeri hesaplanırsa

$$\int_0^x q(t)dt = \int_0^x \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)dx = \left(2x^{1/2} - \frac{2x}{\sqrt{\pi}}\right)\Big|_0^x = 2\sqrt{x} - \frac{2x}{\sqrt{\pi}}$$

elde ederiz. Bulduğumuz değeri (2.39)'da yerine yazarsak

$$\varphi(x,\lambda) = cos\alpha \times cos\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right)$$

$$-cos\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right)$$

$$-\lambda^{-1/2}sin\alpha \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

gerekli düzenlemeler yapılırsa

$$\varphi(x,\lambda) = cos\alpha \times cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$-cos\alpha \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$-\lambda^{-1/2}sin\alpha \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \quad (3.3)$$

olarak bulunur.

**Teorem 3.3:** $\lambda \to \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right) y = 0 \tag{3.1}$$

denkleminin

$$y(\pi, \lambda) = cos\beta, \quad y'(\pi, \lambda) = -sin\beta \qquad \beta \in [0, \pi] \tag{3.4}$$

başlangıç koşullarını sağlayan çözümü

$$\begin{aligned}\psi(x, \lambda) = {} & cos\beta \times cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi} + 1\right)x}{\sqrt{\pi}}\right]\right) \\ & + cos\beta \times sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi} + 1\right)x}{\sqrt{\pi}}\right]\right) \\ & + \lambda^{-1/2} sin\beta \times sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi} + 1\right)x}{\sqrt{\pi}}\right]\right) \\ & + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned} \tag{3.5}$$

ile hesaplanır.

**İspat:** Bir önceki bölümde

$$\begin{aligned}\psi(x, \lambda) = {} & cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ & + cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ & + \lambda^{-1/2} sin\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right) \\ & + O\left(\lambda^{-1/2}\eta(\lambda)\right)\end{aligned} \tag{2.40}$$

olarak bulmuştuk. Buradaki $\int_x^\pi q(t)dt$ değerini hesaplayalım.

$$\int_x^\pi q(t)dt = \int_x^\pi \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)dx = \left(2x^{1/2} - \frac{2x}{\sqrt{\pi}}\right)\Big|_x^\pi$$

$$= \left(2\sqrt{\pi} - \frac{2\pi}{\sqrt{\pi}}\right) - \left(2\sqrt{x} - \frac{2x}{\sqrt{\pi}}\right) = \frac{2x}{\sqrt{\pi}} - 2\sqrt{x}$$

elde ederiz. Bulduğumuz değeri (2.40)'da yerine yazarsak

$$\psi(x,\lambda) = cos\beta \times cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right)$$

$$+cos\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right)$$

$$+\lambda^{-1/2}sin\beta \times sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\left(\frac{2x}{\sqrt{\pi}} - 2\sqrt{x}\right)\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right)$$

gerekli düzenlemeler yapılırsa

$$\psi(x,\lambda) = cos\beta \times cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}\right]\right)$$

$$+cos\beta \times sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}\right]\right)$$

$$+\lambda^{-1/2}sin\beta \times sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.5}$$

elde ederiz.

**Teorem 3.4:** $\lambda \to \infty$ için, (3.2) ve (3.4) koşulları altında

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= 2\lambda^{1/2}cos\alpha \times cos\beta \times \sin\left(\lambda^{1/2}\pi\right) + \lambda^{-1/2}sin\alpha \times sin\beta \times \sin\left(\lambda^{1/2}\pi\right)$$

$$+cos\alpha \times sin\beta \times \left[\sin\left(\lambda^{1/2}\pi\right) - \cos\left(\lambda^{1/2}\pi\right)\right]$$

$$+sin\alpha \times cos\beta \times \left[\sin\left(\lambda^{1/2}\pi\right) + \cos\left(\lambda^{1/2}\pi\right)\right] + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (3.6)$$

ile verilir.

**İspat:** (3.3) ve (3.5) de bulunan $\varphi(x,\lambda)$ ve $\psi(x,\lambda)$ kullanılırsa

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

olduğundan

$$w(\lambda) = 2\lambda^{1/2}cos\alpha \times cos\beta \times \sin\left(\lambda^{1/2}\pi\right) + \lambda^{-1/2}sin\alpha \times sin\beta \times \sin\left(\lambda^{1/2}\pi\right)$$

$$+cos\alpha \times sin\beta \times \left[\sin\left(\lambda^{1/2}\pi\right) - \cos\left(\lambda^{1/2}\pi\right)\right]$$

$$+sin\alpha \times cos\beta \times \left[\sin\left(\lambda^{1/2}\pi\right) + \cos\left(\lambda^{1/2}\pi\right)\right] + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (3.6)$$

elde edilir. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

**Teorem 3.5:**

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \qquad (3.1)$$

denkleminin

$$y(0,\lambda) = cos\alpha, \quad y'(0,\lambda) = -sin\alpha \qquad \alpha \in [0,\pi]$$

ve

$$y(\pi,\lambda) = cos\beta, \quad y'(\pi,\lambda) = -sin\beta \qquad \beta \in [0,\pi]$$

için Green fonksiyonu $(\lambda \to \infty)$

$$G(x,y,\lambda) =$$

$$
= \frac{\begin{bmatrix} \cos\alpha \\ \times \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\cos\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] \end{bmatrix}}
$$

$$
+ O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.7}
$$

olarak bulunur.

**İspat:**

$$
G(x,y,\lambda) = \begin{cases} \dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \le x \le y \le \pi \\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)} + O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0 \le y \le x \le \pi \end{cases}
$$

dir. $0 \le x \le y \le \pi$ için bulduğumuz $\varphi(x,\lambda), \psi(x,\lambda)$ ve $w(\lambda)$'yı $G(x,y,\lambda)$'da yerine yazarsak

$$
G(x,y,\lambda) =
$$

$$
= \frac{\begin{bmatrix} \cos\alpha \\ \times \cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\cos\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -\lambda^{-1/2}\sin\alpha \\ \times \sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} \cos\beta \\ \times \cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\cos\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +\lambda^{-1/2}\sin\beta \\ \times \sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix}}{\begin{bmatrix} 2\lambda^{1/2}\cos\alpha \times \cos\beta \times \sin(\lambda^{1/2}\pi) + \lambda^{-1/2}\sin\alpha \times \sin\beta \times \sin(\lambda^{1/2}\pi) \\ +\cos\alpha \times \sin\beta \times [\sin(\lambda^{1/2}\pi) - \cos(\lambda^{1/2}\pi)] \\ +\sin\alpha \times \cos\beta \times [\sin(\lambda^{1/2}\pi) + \cos(\lambda^{1/2}\pi)] \end{bmatrix}}
$$

$$+\, O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.7}$$

elde ederiz. $0 \le y \le x \le \pi$ için de aynı işlemler yapılır.

Aşağıdaki teoremlerde Dirichlet koşulları altında Green fonksiyonu elde edilecektir.

**Teorem 3.6:**

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \quad (3.1)$$

denkleminin

$$y(0,\lambda) = cos\alpha, \quad y'(0,\lambda) = -sin\alpha \quad \alpha = \frac{\pi}{2} \tag{3.8}$$

başlangıç koşullarını sağlayan çözümü ($\lambda \to \infty$)

**i)**

$$\varphi(x,\lambda) = -\lambda^{-1/2} \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.9}$$

**ii)**

$$\varphi(x,n) = -\frac{1}{(n+1)} \times sin\left(\frac{1}{(n+1)}\left[\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \tag{3.10}$$

ile hesaplanır.

**İspat:**

**i)**Daha önce

$$\varphi(x,\lambda) = -\lambda^{-1/2} \times sin\left(\lambda^{1/2}x - \frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.49}$$

olarak bulmuştuk.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x,\lambda)$’da yerine yazılırsa

$$\varphi(x,\lambda) = -\lambda^{-1/2} sin\alpha \times sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (3.9)$$

elde ederiz.

**ii)** Önceki bölümde

$$\varphi(x,n) = -\frac{1}{(n+1)} \times sin\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (2.46)$$

olarak hesaplamıştık.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x,n)$'de yerine yazılırsa

$$\varphi(x,n) = -\frac{1}{(n+1)} \times sin\left(\frac{1}{(n+1)}\left[\frac{((n+1)^2\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (3.10)$$

elde ederiz.

**Teorem 3.7:**

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \qquad (3.1)$$

denkleminin

$$y(\pi,\lambda) = cos\beta, \quad y'(\pi,\lambda) = -sin\beta \qquad \beta = \frac{\pi}{2} \qquad (3.11)$$

başlangıç koşullarını sağlayan çözümü ($\lambda \to \infty$)

**i)**

$$\psi(x,\lambda) = \lambda^{-1/2} sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (3.12)$$

**ii)**

$$\psi(x,n)=\frac{1}{(n+1)}\times sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (3.13)$$

ile bulunur.

**İspat:**

**i)** Bir önceki bölümde

$$\psi(x,\lambda)=\lambda^{-1/2}sin\left(\lambda^{1/2}(\pi-x)-\frac{1}{2\lambda^{1/2}}\int_x^\pi q(t)dt\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (2.53)$$

olarak bulmuştuk. $\int_x^\pi q(t)dt$ hesaplanır ve $\psi(x,\lambda)$'da yerine yazılırsa

$$\psi(x,\lambda)=\lambda^{-1/2}sin\left(\frac{1}{\lambda^{1/2}}\left[\sqrt{x}-\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (3.12)$$

elde ederiz.

**ii)** Daha önce

$$\psi(x,n)=\frac{1}{(n+1)}\times sin\left((n+1)(\pi-x)-\frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (2.54)$$

olarak hesaplamıştık.

$\int_x^\pi q(t)dt$ hesaplanır ve $\psi(x,n)$'de yerine yazılırsa

$$\psi(x,n)=\frac{1}{(n+1)}\times sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (3.13)$$

elde ederiz.

**Teorem 3.8:** $\lambda \to \infty$, (3.8) ve (3.11) koşulları altında

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

$$= \lambda^{-1/2}\sin\left(\lambda^{1/2}\pi\right) + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (3.14)$$

olarak bulunur.

**İspat:** (3.9) ve (3.12)'de bulunan $\varphi(x,\lambda)$ ve $\psi(x,\lambda)$ kullanılırsa

$$w(\lambda) = w_x(\varphi,\psi) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

olduğundan

$$w(\lambda) = \lambda^{-1/2}\sin\left(\lambda^{1/2}\pi\right) + O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (3.14)$$

elde edilir. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

**Teorem 3.9:**

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \qquad (3.1)$$

denkleminin

$$y(0,\lambda) = 0, \quad y(\pi,\lambda) = 0$$

sınır değer problemi için Green fonksiyonu $(\lambda \to \infty)$

**i)**

$$G(x,y,\lambda) = \frac{\left[\begin{array}{c} sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ \times \\ sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{\left(\lambda\sqrt{\pi}+1\right)y}{\sqrt{\pi}}\right]\right) \end{array}\right]}{\lambda^{1/2}\sin(\lambda^{1/2}\pi)} + O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (3.15)$$

**ii)**

$$G(x,y,n)=\frac{1}{(n+1)}sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$\times sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{y}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)y}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-3}\eta(n)\right) \tag{3.16}$$

ile verilir.

**İspat:**

**i)**

$$G(x,y,\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}+O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0\le x\le y\le\pi\\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}+O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0\le y\le x\le\pi\end{cases}$$

dir. $0\le x\le y\le\pi$ için bulduğumuz $\varphi(x,\lambda),\psi(x,\lambda)$ ve $w(\lambda)$'yı $G(x,y,\lambda)$'da yerine yazarsak

$$G(x,y,\lambda)=\frac{\left[\begin{array}{c}sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)\\ \times\\ sin\left(\lambda^{1/2}\pi+\frac{1}{\lambda^{1/2}}\left[\sqrt{y}-\frac{\left(\lambda\sqrt{\pi}+1\right)y}{\sqrt{\pi}}\right]\right)\end{array}\right]}{\lambda^{1/2}\sin(\lambda^{1/2}\pi)}+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.15}$$

elde ederiz. $0\le y\le x\le\pi$ için de aynı işlemler yapılır.

**ii)** Bulduğumuz $G(x,y,\lambda)$'ya reversion uygularsak

$$G(x,y,n)=\frac{1}{(n+1)}sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$\times sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{y}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)y}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-3}\eta(n)\right) \tag{3.16}$$

elde ederiz.

Aşağıdaki teoremlerde Neumann koşulları altında Green fonksiyonu elde edilecektir.

**Teorem 3.10:**

$$y''+\left(\lambda-\left(x^{-1/2}-\frac{2}{\sqrt{\pi}}\right)\right)y=0 \tag{3.1}$$

denkleminin

$$y(0,\lambda)=cos\alpha,\quad y'(0,\lambda)=-sin\alpha \qquad \alpha=0 \tag{3.17}$$

başlangıç koşullarını sağlayan çözümü

**i)**

$$\varphi(x,\lambda)=cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)-sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.18}$$

**ii)**

$$\varphi(x,n)=cos\left(\frac{1}{(n+1)}\left[\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)$$

$$-sin\left(\frac{1}{(n+1)}\left[\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)+O\left(n^{-2}\eta(n)\right). \tag{3.19}$$

**İspat:**

**i)** Daha önce

$$\varphi(x,\lambda)=cos\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)-sin\left(\lambda^{1/2}x-\frac{1}{2\lambda^{1/2}}\int_0^x q(t)dt\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.65}$$

olarak bulmuştuk.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x,\lambda)$'da yerine yazılırsa

$$\varphi(x,\lambda) = cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) - sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.18}$$

elde ederiz.

**ii)** Önceki bölümde

$$\varphi(x,n) = cos\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right)$$

$$-sin\left((n+1)x - \frac{1}{2(n+1)}\int_0^x q(t)dt\right) + O\left(n^{-2}\eta(n)\right) \tag{2.66}$$

olarak hesaplamıştık.

$\int_0^x q(t)dt$ hesaplanır ve $\varphi(x,n)$'de yerine yazılırsa

$$\varphi(x,n) = cos\left(\frac{1}{(n+1)}\left[\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right)$$

$$-sin\left(\frac{1}{(n+1)}\left[\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) + O\left(n^{-2}\eta(n)\right) \tag{3.19}$$

elde ederiz.

**Teorem 3.11:** $\lambda \to \infty$

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right)y = 0 \tag{3.1}$$

denkleminin

$$y(\pi,\lambda) = cos\beta, \quad y'(\pi,\lambda) = -sin\beta, \quad \beta = 0 \tag{3.20}$$

başlangıç koşullarını sağlayan çözümü

**i)**

$$\psi(x,\lambda) = cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.21}$$

**ii)**

$$\psi(x,n) = cos\left((n+1)\pi + \frac{1}{(n+1)}\left[\sqrt{x} - \frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+sin\left((n+1)\pi + \frac{1}{(n+1)}\left[\sqrt{x} - \frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \tag{3.22}$$

ile bulunur.

**İspat:**

**i)** Bir önceki bölümde

$$\psi(x,\lambda) = cos\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right)$$

$$+sin\left(\lambda^{1/2}(\pi - x) - \frac{1}{2\lambda^{1/2}}\int_x^{\pi} q(t)dt\right) + O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{2.72}$$

olarak bulmuştuk. $\int_x^{\pi} q(t)dt$ hesaplanır ve $\psi(x,\lambda)$'da yerine yazılırsa

$$\psi(x,\lambda) = cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{x} - \frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+sin\left(\lambda^{1/2}\pi+\frac{1}{\lambda^{1/2}}\left[\sqrt{x}-\frac{\left(\lambda\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)+O\left(\lambda^{-1/2}\eta(\lambda)\right) \qquad (3.21)$$

elde ederiz.

**ii)** Daha önce

$$\psi(x,n)=cos\left((n+1)(\pi-x)-\frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)$$

$$+sin\left((n+1)(\pi-x)-\frac{1}{2(n+1)}\int_x^\pi q(t)dt\right)+O\left(n^{-2}\eta(n)\right) \qquad (2.73)$$

olarak hesaplamıştık. $\int_x^\pi q(t)dt$ hesaplanır ve $\psi(x,n)$'de yerine yazılırsa

$$\psi(x,n)=cos\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+sin\left((n+1)\pi+\frac{1}{(n+1)}\left[\sqrt{x}-\frac{\left((n+1)^2\sqrt{\pi}+1\right)x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-2}\eta(n)\right) \qquad (3.22)$$

elde ederiz.

**Teorem 3.12:** $\lambda\to\infty$, (3.17) ve (3.20) koşulları altında

$$w(\lambda)=w_x(\varphi,\psi)=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda)$$

$$=2\lambda^{1/2}\sin\left(\lambda^{1/2}\pi\right)+O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (3.23)$$

olarak bulunur.

**İspat:** (3.18) ve (3.21)'de bulunan $\varphi(x,\lambda)$ ve $\psi(x,\lambda)$ kullanılırsa

$$w(\lambda)=w_x(\varphi,\psi)=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda)$$

olduğundan

$$w(\lambda)=2\lambda^{1/2}\sin\left(\lambda^{1/2}\pi\right)+O\left(\lambda^{-1}\eta(\lambda)\right) \qquad (2.79)$$

şeklinde hesaplarız. Dikkat edilirse $w(\lambda)$ eşitliğimiz $q(t)$ fonksiyonuna bağlı değildir.

**Teorem 3.13:**

$$y'' + \left(\lambda - \left(x^{-1/2} - \frac{2}{\sqrt{\pi}}\right)\right) y = 0 \tag{3.1}$$

denkleminin

$$y'(0,\lambda) = 0, \quad y'(\pi,\lambda) = 0$$

sınır değer problemi için Green fonksiyonu $(\lambda \to \infty)$

**i)**

$$G(x,y,\lambda) =$$

$$\frac{\begin{bmatrix} cos\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \\ -sin\left(\frac{1}{\lambda^{1/2}}\left[\frac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}} - \sqrt{x}\right]\right) \end{bmatrix} \times \begin{bmatrix} cos\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \\ +sin\left(\lambda^{1/2}\pi + \frac{1}{\lambda^{1/2}}\left[\sqrt{y} - \frac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right) \end{bmatrix}}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.24}$$

**ii)**

$$G(x,y,n) = cos\left((n+1)\pi + (n+1)(x-y) - \frac{1}{(n+1)}\left[\sqrt{x} - \sqrt{y} + \frac{y-x}{\sqrt{\pi}}\right]\right)$$

$$\times\, sin\left((n+1)(\pi - y) - (n+1)x + \frac{1}{(n+1)}\left[\sqrt{x} + \sqrt{y} - \frac{y+x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-3}\eta(n)\right) \tag{3.25}$$

olarak hesaplanır.

**İspat:**

**i)**

$$G(x,y,\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(y,\lambda)}{w(\lambda)}+O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0\le x\le y\le\pi\\ \dfrac{\varphi(y,\lambda)\psi(x,\lambda)}{w(\lambda)}+O\left(\lambda^{-1/2}\eta(\lambda)\right), & 0\le y\le x\le\pi\end{cases}$$

dir. $0\le x\le y\le\pi$ için bulduğumuz $\varphi(x,\lambda),\psi(x,\lambda)$ ve $w(\lambda)$'yı $G(x,y,\lambda)$'da yerine yazarsak

$$G(x,y,\lambda)=$$

$$\frac{\begin{bmatrix}cos\left(\dfrac{1}{\lambda^{1/2}}\left[\dfrac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)\\ -sin\left(\dfrac{1}{\lambda^{1/2}}\left[\dfrac{(\lambda\sqrt{\pi}+1)x}{\sqrt{\pi}}-\sqrt{x}\right]\right)\end{bmatrix}\times\begin{bmatrix}cos\left(\lambda^{1/2}\pi+\dfrac{1}{\lambda^{1/2}}\left[\sqrt{y}-\dfrac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right)\\ +sin\left(\lambda^{1/2}\pi+\dfrac{1}{\lambda^{1/2}}\left[\sqrt{y}-\dfrac{(\lambda\sqrt{\pi}+1)y}{\sqrt{\pi}}\right]\right)\end{bmatrix}}{2\lambda^{1/2}\sin(\lambda^{1/2}\pi)}$$

$$+O\left(\lambda^{-1/2}\eta(\lambda)\right) \tag{3.24}$$

elde ederiz. $0\le y\le x\le\pi$ için de aynı işlemler yapılır.

**ii)** Bulduğumuz $G(x,y,\lambda)$'ya reversion uygularsak

$$G(x,y,n)=cos\left((n+1)\pi+(n+1)(x-y)-\frac{1}{(n+1)}\left[\sqrt{x}-\sqrt{y}+\frac{y-x}{\sqrt{\pi}}\right]\right)$$

$$\times sin\left((n+1)(\pi-y)-(n+1)x+\frac{1}{(n+1)}\left[\sqrt{x}+\sqrt{y}-\frac{y+x}{\sqrt{\pi}}\right]\right)$$

$$+O\left(n^{-3}\eta(n)\right) \tag{3.26}$$

elde ederiz.

## 4.SONUÇLAR

1) $y'' + (\lambda - q)y = 0$ (1) probleminin $y(0,\lambda) = cos\alpha,\ \ y'(0,\lambda) = -sin\alpha,$ $\alpha \in [0,\pi]$ başlangıç değer koşullarını sağlayan çözümü Teorem 2.1 ile verildi. Uygun notasyonlar yardımıyla

$$\varphi(x,\lambda) = G_0\big(E_0, C_0, S_0, T(0,\lambda), S(0,\lambda)\big)$$

olarak ifade edildi.

2) (1) probleminin $y(\pi,\lambda) = cos\beta,\ \ y'(\pi,\lambda) = -sin\beta,\ \ \beta \in [0,\pi]$ başlangıç değer koşullarını sağlayan çözümü Teorem 2.2 ile verildi. Notasyonlar yardımıyla

$$\psi(x,\lambda) = F_\pi\big(E_\pi, C_\pi, S_\pi, T(\pi,\lambda), S(\pi,\lambda)\big)$$

ifade edildi.

3) 1) ve 2)'de gerekli $E_0, C_0, S_0$ ve $E_\pi, C_\pi, S_\pi$ değerleri Teorem 2.3'te hesaplandı.

4) 3)'de hesaplanan değerler yardımıyla $\varphi(x,\lambda)$ ve $\psi(x,\lambda)$ çözümleri $\lambda$'ya bağlı asimptotik olarak Teorem 2.4'de elde edildi.

5) Green fonksiyonu için gerekli olan Wronskian $w(\lambda)$ Teorem 2.5'te hesaplandı ve $G(x,y,\lambda)$ değeri Teorem 2.6'a verildi.

6) Özel olarak Dirichlet koşullarının sağlanması durumu $\left(\alpha = \frac{\pi}{2}\right)$ durumu ele alınarak $\varphi(x,\lambda)$ için asimptotik çözüm bulundu, daha sonra $\lambda_n$ için asimptotik tahminler kullanılarak $\varphi(x,n)$ çözümü Teorem 2.7'de elde edildi.

7) Benzer şekilde $\beta = \frac{\pi}{2}$ için $\psi(x,\lambda)$ ve daha sonra $\psi(x,n)$ çözümleri Teorem 2.8'de verildi.

8) Dirichlet sınır koşulları için Wronskian $w(\lambda)$ Teorem 2.9'da hesaplandı.

9) Wronskian kullanılarak Dirichlet sınır koşulları durumunda $G(x, y, \lambda)$ asimptotik olarak Teorem 2.10'da verildi.

10) (6-9)'da elde edilen sonuçlara benzer olarak $\alpha = 0$ ve $\beta = 0$ için Neumann sınır koşulları durumu incelenerek $G(x, y, \lambda)$ Green fonksiyonu Teorem 2.14'de ifade edildi.

11) Son olarak yukarıdaki sonuçlar $q(x) = x^{-1/2} - \frac{2}{\sqrt{\pi}}$ potansiyel fonksiyonuna uygulandı.

## 5.ÖNERİLER

1) Elde edilen asimptotik sonuçlarda $q(x)$ potansiyel fonksiyonu integrallenebilen bir fonksiyon olarak kabul edildi. Potansiyel fonksiyonunun singüler nokta içermesi durumu için de asimptotik çözümler araştırılabilir.

2) Sınır koşulları her iki uç noktada ya da bir uçta $\lambda$'ya bağlı farklı formlarda kabul edilerek çözümler araştırılabilir.

3) Potansiyel fonksiyonunun sınırlı salınımlı bir fonksiyon olması durumu incelenebilir.

4) Asimptotik çözümler için hata terimi geliştirilebilir.

## 6.KAYNAKLAR


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## ÖZGEÇMİŞ

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