ON ROW CO-DIVISORS IN REGULAR MATRICES

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ABSTRACT

This study is about the solution of the equation XA=B, similar to the matrix division revealed by obtaining the solution of the matrix equation AX=B in 2010. This study is about the solution of the equation XA=B, similar to the matrix division revealed by obtaining the solution of the matrix equation AX=B in 2010. The solution of the AX=B equation is based on the displacement of the columns. Taking this into account, the definition of division in matrices is made. The definition of division is not contradict with matrix multiplication. This provided many benefits. For example, the solution of $n-$ times systems of type $m\times n$ is easily obtained. Taking this gain into account, the solution of $m\times n$, $n-$ times system of equations in the form of XA=B is investigated in study. As it turns out, this solution is based on the displacement of rows. The theoretical expression of this approach is given. The features between the previously given partition and the partition were examined. Necessary comparisons were made. The rapid results are obtained in the local and even holistic comparison of the matrices obtained from the systems. Therefore, The row co-divisior is expected to make a significant contribution to the comparison of technological solutions. On the contrary, this approach allows observing mass solutions, variation between local solutions. Also, the link between the transpose, the determinant, division, and row change of a matrix is obtained in the study. This sduty is introduced definitions, features, and lemmas to the new literature.

Anahtar Kelimeler : row co-divisor, column co-divisor, solution equations, division.

1. INTRODUCTION

Let us start with the introduction and selection of the elements for the statements in this section. Let F be a field and $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^+\right\}$. Also, the transpose of $A\in M_n(F)$ is denoted by A^T. And
The definition that will form the basis for the solution of the $XA=B$ linear equation is given below.

Definition 1.1. Let A and B be two regular square matrices of order n. The determinant of the new matrix obtained by writing the i^{th} row of the matrix A on the j^{th} row of the matrix B is called the *co-divisor by row* of the matrix A by the row on the matrix B. It is denoted by $\underset{ij}{AB}$. Their number is n^2. The matrix co-divisor by row is $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$.

column-by-column co-divisors are denoted by ${}_{B}^{A}i_{j}$ and column-by-column co-divisors matrix as $\left[\left({}_{B}^{A}i_{j}\right)_{ij}\right]$ (Keleş, 2010,2015, 2018, 2022).

The following lemmas and theorems that have been proved before are given.

Lemma 1.2. Let F be a field, $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^{+}\right\}$ and $A,B\in M_n(F)$. Then,

$$\frac{1}{|A|}\left[\left(\underset{ij}{AB}\right)_{ij}\right]=\left(\frac{B^T}{A^T}\right)^T \text{ (Keleş, 2022).}$$

Theorem 1.3. Let F be a field, $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^{+}\right\}$ and $A,B\in M_n(F)$. Then,

$$\frac{A}{B}=\frac{B^T}{A^T}.$$

Theorem 1.4. Let F be a field, $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^{+}\right\}$ and $A,B\in M_n(F)$. Then, solution of the linear matrix equation $XA=B$;

$$X=\left(\frac{B^T}{A^T}\right)^T.$$

Lemma 1.5. Let F be a field, $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^{+}\right\}$ and $A,B,C\in M_n(F)$, where $|C|\neq 0$. Then,

i. $\left[\left({}_{B}^{A}i_{j}\right)_{ij}\right]=\frac{1}{|C|}\left[\left({}_{B}^{C}i_{j}\right)_{ij}\right]\left[\left({}_{C}^{A}i_{j}\right)_{ij}\right]$.

ii. $\left[\left({}_{B}^{A}i_{j}\right)_{ij}\right]=A\left[\left({}_{B}^{I}i_{j}\right)_{ij}\right]$.

The theorem, whose proof was given in previous studies, is given below.

Theorem 1.6. Let $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \square^{+}\right\}$ be and $C_{ij}(A)=(-1)^{i+j}M_{ij}(A)$ be, where $M_{ij}(A)$ is the minor of a_{ij}. If $A\in M_n(F)$ with $|A|\neq 0$, let I *or* I_n be the unit matrix. Then

i. ${}_{A}^{I}i_{j}=C_{ij}(A)$.

ii. ${}_{I}^{A}i_{j}=a_{ij}$ where $i,j=1,\cdots,n$.

iii. ${}_{A}^{A}i_{j}=\begin{cases}|A|, if\ \ i=j\\ 0, if\ \ i\neq j\end{cases}$

2. ROW CO-DIVISORS IN REGULAR MATRICES

The fact of matrices provide the Pololoid structure is considered in this section.

In this section, the similarities, equality or differences between $\left[\left(\begin{matrix}A\\B\end{matrix}i\,j\right)_{ij}\right]$ and $\left[\left(AB\right)_{ij}\right]$, which form the basis of the solution of equations, $XA=B$ and $AX=B$ formed by the square matrices of A and B are examined.

Example 2.1. The row and column co-divisors of second-order square matrices $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ and $B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$ are given below.

$$\underset{11}{AB}=\begin{vmatrix}a_{11} & a_{12}\\ b_{21} & b_{22}\end{vmatrix}=a_{11}b_{22}-a_{12}b_{21}$$

$$\underset{12}{AB}=\begin{vmatrix}b_{11} & b_{12}\\ a_{11} & a_{12}\end{vmatrix}=b_{11}a_{12}-a_{11}b_{12}$$

$$\underset{21}{AB}=\begin{vmatrix}a_{21} & a_{22}\\ b_{21} & b_{22}\end{vmatrix}=a_{21}b_{22}-b_{21}a_{22}$$

$$\underset{22}{AB}=\begin{vmatrix}b_{11} & b_{12}\\ a_{21} & a_{22}\end{vmatrix}=b_{11}a_{22}-a_{21}b_{12}$$

$$\begin{matrix}A\\B\end{matrix}1_1=\begin{vmatrix}a_{11} & b_{12}\\ a_{21} & b_{22}\end{vmatrix}=a_{11}b_{22}-a_{21}b_{12}$$

$$\begin{matrix}A\\B\end{matrix}1_2=\begin{vmatrix}b_{11} & a_{11}\\ b_{21} & a_{21}\end{vmatrix}=b_{11}a_{21}-b_{21}a_{11}$$

$$\begin{matrix}A\\B\end{matrix}2_1=\begin{vmatrix}a_{12} & b_{12}\\ a_{22} & b_{22}\end{vmatrix}=a_{12}b_{22}-a_{22}b_{12}$$

$$\begin{matrix}A\\B\end{matrix}2_2=\begin{vmatrix}b_{11} & a_{12}\\ b_{21} & a_{22}\end{vmatrix}=b_{11}a_{22}-a_{12}b_{21}$$

$$\left[\left(AB\right)_{ij}\right]=\begin{bmatrix}a_{11}b_{22}-a_{12}b_{21} & b_{11}a_{12}-a_{11}b_{12}\\ a_{21}b_{22}-b_{21}a_{22} & b_{11}a_{22}-a_{21}b_{12}\end{bmatrix}$$

$$\left[\left(\begin{matrix}A\\B\end{matrix}i\,j\right)_{ij}\right]=\begin{bmatrix}a_{11}b_{22}-a_{21}b_{12} & b_{11}a_{21}-b_{21}a_{11}\\ a_{12}b_{22}-a_{22}b_{12} & b_{11}a_{22}-a_{12}b_{21}\end{bmatrix}$$

Lemma 2.2. Let $A,B\in M_n\left(F\right)$ be regular matrices. Then

$$\left[\left(\begin{matrix}A^T\\B^T\end{matrix}i\,j\right)_{ij}\right]=\left[\left(AB\right)_{ij}\right]^T.$$

Proof. Let the regular matrices $A,B\in M_2\left(F\right)$ be given

$$\left[\left(\begin{matrix}A^T\\B^T\end{matrix}i\ j\right)_{ij}\right]=\begin{bmatrix}a_{11}b_{22}-a_{12}b_{21} & a_{21}b_{22}-a_{22}b_{21}\\ a_{12}b_{11}-a_{11}b_{12} & a_{22}b_{11}-a_{21}b_{12}\end{bmatrix}$$

$$\left[\left(\underset{ij}{AB}\right)_{ij}\right]^T==\begin{bmatrix}a_{11}b_{22}-a_{12}b_{21} & a_{21}b_{22}-a_{22}b_{21}\\ a_{12}b_{11}-a_{11}b_{12} & a_{22}b_{11}-a_{21}b_{12}\end{bmatrix}.$$

Lemma 2.3. Let F be a field, $M_n\left(F\right)=\left\{\left[a_{ij}\right]_n\middle|a_{ij}\in F,n\in\square^+\right\}$ and $A,B,C\in M_n\left(F\right)$, where $\left|C\right|\neq0$. Then,

i. $$\left[\left(\underset{ij}{AB}\right)_{ij}\right]^T=\frac{1}{\left|C\right|}\left[\left(\begin{matrix}C\\B^T\end{matrix}i_j\right)_{ij}\right]\left[\left(\begin{matrix}A^T\\C\end{matrix}i_j\right)_{ij}\right].$$

ii. $$\left[\left(\underset{ij}{AB}\right)_{ij}\right]^T=A^T\left[\left(\begin{matrix}I\\B^T\end{matrix}i_j\right)_{ij}\right].$$

Proof. Let the regular matrices $A,B,C\in M_2\left(F\right)$ be given, where $\left|C\right|\neq0$. Its proof is obtained by Lemma 1.5 given in the Introduction and Lemma 2.2 in this second.

Theorem 2.4. Let $M_n\left(F\right)=\left\{\left[a_{ij}\right]_n\middle|a_{ij}\in F,n\in\square^+\right\}$ be and $C_{ij}\left(A\right)=\left(-1\right)^{i+j}M_{ij}\left(A\right)$ be, where $M_{ij}\left(A\right)$ is the minor of a_{ij}. If $A\in M_n\left(F\right)$ with $\left|A\right|\neq0$, let I or I_n be the unit matrix. Then

1. $\begin{matrix}I_n\\A^T\end{matrix}i_j=\underset{ji}{I_nA}$.
2. $\underset{ji}{AI_n}=a_{ji}$, where $i=1,\cdots,n$ and $j=1,\cdots,n$.
3. $\underset{ij}{AA}=\begin{cases}\left|A\right|,i=j\\0,i\neq j\end{cases}$

Proof. For a matrix A that satisfies the hypothesis of the theorem

1. $\begin{matrix}I_n\\A^T\end{matrix}i_j=C_{ij}\left(A^T\right)=\underset{ji}{I_nA}$.
2. $\begin{matrix}A^T\\I_n\end{matrix}i_j=a_{ji}=\underset{ji}{AI_n}$.
3. $\begin{matrix}A\\A\end{matrix}i_j=\underset{ij}{AA}=\begin{cases}\left|A\right|,i=j\\0,i\neq j\end{cases}$.

Theorem 2.5. If $XA=B,AX=B$, then

$$\left(\frac{B^T}{A^T}\right)^T=B\left(\frac{A}{B}\right).$$

Proof. If $XA=B,AX=B$ then,

$$AX = B \Rightarrow X = \frac{B}{A}, XA = B \Rightarrow X = \left(\frac{B^T}{A^T}\right)^T$$

$$XA = B = \left(\frac{B^T}{A^T}\right)^T \left(\frac{B}{A}\right) = B,$$

$$\left(\frac{B^T}{A^T}\right)^T = B\left(\frac{A}{B}\right).$$

3. RESULTS AND DISCUSSIONS

Comparison of the results obtained by multiplying two matrices is aimed and concluded. Issues related to the unknowns obtained by right multiplication are still among the subjects that need to be studied.

4. REFERENCES

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