On Some Properties of Poloids and Mappings

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Introduction

This section is about the poloid created by introducing the new property to the monoid. The concept of monoid, which has a great place in algebraic theory, is the important categorical expression. The expression of monoids in a cryptic manner is made meaningful to new properties. This cryptic categorical structure is made even more evident with the concept of poloid. In particular, the monoid is extended to the poloid with the condition (P4). The poloid structure is gained even more meaning. The new results presented on poloid provide additional contributions to algebraic structures such as goup, ring, and field. Better expression of these structures allows for a better understanding of categorical structures. Obtaining partial transformation systems gives us the opportunity to recognize similar structures. Jonsson gave partial property of the poloid structure in 2018. The structure is made even more prominent in 2022.

The (P4) feature covers the definition given by Jonsson.

Definition 1. Let $(G,*)$ be group. $(G,*)$ is called a poloid such that for every $x \in G$ there is a unique $x^{-1} \in G$ such that $x * x^{-1}$ and $x^{-1} * x$ are defined and units.

Definition 2. A group is a set P equipped with a binary operation $*: P \times P \to P$ that associates an element $a * b \in P$ to every pair of elements $a, b \in P$, and having the following properties: $*$ is associative, has an identity element $e \in P$, and every element in P is invertible More explicitly, this means that the following equations hold for all $a, b, c, d \in P$:

$P1.\ a * (b * c) = (a * b) * c$, (associativity).

$P2.\ a * e = e * a$, (identity).

$P3.$ For every $a \in P$, there is some $a^{-1} \in P$ such that $a * a^{-1} = a^{-1} * a = e$ (inverse).

$P4.$ For every $a \in P \setminus \{e\}$, there are some $b, c, d \in P \setminus \{e\}$ such that $b * c = c * d = a$ with $b \neq d$ (escort).

A set P together with an operation $*: P \times P \to P$ and satisfying only conditions (P1), (P2), (P3) and (P4) is called a poloid. It is denoted by $(P,*)$.

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Properties of Poloids and Mappings

In this chapter, besides the operational properties of some matrices, the subject of mappings is covered.

The following lemma explains this situation.

Lemma 1. If $(P,*)$ is a poloid then for every $x \in P$ there is a unique $x^{-1} \in P$ such that $x * x^{-1}$ and $x^{-1} * x$ are defined.

Proof. If $(P,*)$ is a poloid then for every $a \in P \setminus \{e\}$, there are some $b,c,d \in P \setminus \{e\}$ such that $b*c = c*d = a$ with $b \neq d$.

$$(b*c)*x = (c*d)*x = a*x = e, \text{ where } x \in P$$

$b*c = c*d = a$ by (P3).

$$b*c = c*d = a \Rightarrow x = (b*c)^{-1} = c^{-1} * b^{-1}$$

$$x = (c*d)^{-1} = d^{-1} * c^{-1},$$

And

$$x := a^{-1}.$$

a^{-1} element is also uniqe since $b,c,d \in P \setminus \{e\}$ are uniqe elements.

Note: Each element of the unit different poloid is obtained from the elements one common and one different of group.

Property 2. Let $(P,*)$ be a poloid. If all $x \in P \setminus \{e\}$ $x = x_2 * x_1 = x_1 * x_3$ then

$$x^{-1} = x_1^{-1} * x_2^{-1} = x_3^{-1} * x_1^{-1}.$$

Proof. For all $x \in P \setminus \{e\}$ $x = x_2 * x_1 = x_1 * x_3$,

$$x * x^{-1} = x_2 * x_1 * x^{-1} = e$$

$$x^{-1} = x_1^{-1} * x_2^{-1},$$

And

$$x * x^{-1} = x_1 * x_3 * x^{-1} = e$$

$$x^{-1} = x_3^{-1} * x_1^{-1},$$

$$x^{-1} = x_1^{-1} * x_2^{-1} = x_3^{-1} * x_1^{-1}, \text{ where } x_2^{-1} \neq x_3^{-1}.$$

Let $f : P \to P$ be a mapping satisfying the following properties

i. For every $y \in P$ there exist $x \in P$ such taht $y = f(x)$.

ii. If every two elements $x_1, x_2 \in P$ there exist $y_1, y_2 \in P$ such that $y_1 = f(x_1), y_2 = (x_2)$ then $y_1 = y_2$.

iii. For any two $x_1, x_2 \in P$ $f(x_1 * x_2) = f(x_1) * f(x_2)$.

$$f : P \to P$$

$$x \to y = f(x).$$

The f transform that satisfies the three conditions above is called the poloid mapping.

If $f(P) = \{x | f(x) = x, x \in P\}$, then $(f(P), *)$ is a poloid. It is clearly.

Lemma 3. Let f be a poloid mapping. The following situations hold.

i. For every $x \in P$ there exist $x_1, x_2 \in P \setminus \{e\}$ such taht $f(x) = f(x_1) * f(x_2)$.

ii. For every two $x_1, x_2 \in P \setminus \{e\}$ there exist $x_3 \in P \setminus \{e\}$ such taht $f(x_2) * f(x_1) = f(x_1) * f(x_3)$, where $f(x_2) \neq f(x_3)$.

iii. For every $x \in P$ there exist $x_1, x_2 \in P \setminus \{e\}$ such taht $f^{-1}(x) = f^{-1}(x_2) * f^{-1}(x_1)$.

iv. $f(x^{-1}) = f^{-1}(x)$ for every $x \in P$.

v. $f(e) = e$

Proof. Let $(P, *)$ be a poloid and f a poloid mapping.

i. For every $x \in P$ there exist $x_1, x_2 \in P \setminus \{e\}$ such taht $x = x_1 * x_2$ by (P4).

If a mapping f then

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2).$$

ii. For every $x \in P$ there exist $x_1, x_2, x_3 \in P \setminus \{e\}$ such taht $x = x_2 * x_1 = x_1 * x_3$ by (P4).

$$f(x_2) * f(x_1) = f(x_1) * f(x_3)$$

$f(x_2) \neq f(x_3)$ because $x_2 \neq x_3$.

iii. For every $x \in P$ there exist $x_1, x_2 \in P \setminus \{e\}$ such taht $f(x) = f(x_1) * f(x_2)$,

$$\begin{aligned} e = y * y^{-1} &= f(x) * f^{-1}(x) \\ &= f(x_1) * f(x_2) * f(x_2^{-1}) * f(x_1^{-1}) \\ &= f(x_1) * f(x_2) * f^{-1}(x_2) * f^{-1}(x_1) \end{aligned}$$

$$y^{-1} = f^{-1}(x) = f^{-1}(x_2) * f^{-1}(x_1).$$

iv. If $x \in P$ then

$$e = f(x) * f^{-1}(x) = f(x) * f(x^{-1}).$$

v.

$$\begin{aligned} e &= f(x) * f^{-1}(x) \\ &= f(x) * f(x^{-1}) \\ f(e) &= f(x * x^{-1}) = e. \end{aligned}$$

Lemma 4. Let f be a poloid mapping. The following situations hold.

i. For every $x \in P$ there exist $x_1, x_2 \in P \setminus \{e\}$ such taht

$$(f \circ f)(x) = (f \circ f)(x_1) * (f \circ f)(x_2).$$

ii. If $x = e$ then

$$\left(\underbrace{f \circ f \circ \cdots \circ f}_{n-times} \right)(e) = e.$$

Proof. Let $(P, *)$ be a poloid and f a poloid mapping.

i. It is clearly.

ii. If $x = e$ then

$$\begin{aligned} \left(\underbrace{f \circ f \circ \cdots \circ f}_{n-times} \right)(e) &= \left(\underbrace{f \circ f \circ \cdots \circ f}_{(n-1)-times} \right)(f(e)) \\ &= \left(\underbrace{f \circ f \circ \cdots \circ f}_{(n-1)-times} \right)(e) \\ &\vdots \\ &= f(e) = e. \end{aligned}$$

Definition 5. Let $(P, *)$ be a poloid and $x, x_1 \in P$. x_1 is called divides x if $x = x_1 * x_2$, where $x_2 \in P$. They are denoted $x_1 | x$ and $x_2 = \frac{x}{x_1}$. Each of the two elements of the poloid divides the other. x_1 is called not divide x if $x \neq x_1 * x_2$. It is denoted $x_1 \nmid x$.

Note. The set of regular matrices is given as the example of poloid in our previous studies.

The set of all regular matrices of order $n \in \mathbb{Z}^+$ over a field F is denoted by

$$M_n(F) = \left\{ \left[a_{ij} \right]_n \middle| a_{ij} \in F, n \in \mathbb{Z}^+ \right\}$$

The transpose of $A \in M_n(F)$ is denoted by A^T.

If $M_n(\mathbb{C})$ and considering the elementary operations in the matrices $M_n(\mathbb{C})$, the following property is given without proof.

Example 6. If $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & \frac{6}{5} \\ \frac{1}{2} & \frac{7}{10} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = B.$$

Thus, $A \mid B$.

Example 7. If $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 8 \\ 24 & 10 \end{bmatrix},$$

$A \not\mid B$ that $AC \neq B$.

Property 8. For all $A, B, C \in M_n(\mathbb{C})$ the followings hold.

i. If $A, B \in M_n(\mathbb{C})$ then $A \mid B$, $C = \frac{B}{A}$ and $B \mid A$, $C = \frac{A}{B}$.

ii. If $A \mid B$ then $A = \left(\frac{B^T}{C^T} \right)^T$.

iii. If $A \not\mid B$ then $B = AC + D, C = \frac{B - D}{A}$.

Lemma 9. For any $A, B \in M_n(\mathbb{C})$ If $A \not\mid B$, then there exist $D_1 \in M_n(\mathbb{C})$ such that $B = A(C + D_1)$.

Proof. For any $A, B \in M_n(\mathbb{C})$ If $A \not\mid B$, then

$$B = AC + D,$$

$$A \mid D \text{ and } D = AD_1, \text{ where } D_1 \in M_n(\square),$$

$$B = AC + AD_1,$$

$$B = A(C + D_1).$$

Corollary. If $A \nmid B$ then there exist two unique $C, D_1 \in M_n(\square)$ such that $A = \left(\frac{B^T}{(C + D_1)^T} \right)^T$.

Property 10. Let $(P, *)$ be a poloid. There are infinite divisors every element different from unit.

Proof. Let $(P, *)$ be a poloid. For every $a \in P \setminus \{e\}$, there are some $b, c, d \in P \setminus \{e\}$ such that $a = b * c$. There are $b_1, s_1 \in P \setminus \{e\}$ such that $b = b_1 * s_1$ for $b \in P \setminus \{e\}$.

$$a = b * c$$

$$a = b_n * (s_n * c), \text{ where } n \in \square^+.$$

$$b_n \mid a.$$

Lemma 11. Let $(P, *)$ be a poloid and f a poloid mapping.

i. If $x, x_1, x_2 \in P$ then

$$f(x_1) \mid f(x).$$

ii. $f^{-1}(x) = \frac{e}{f(x)}$.

Proof. Let $(P, *)$ be a poloid and f a poloid mapping.

i. If $x, x_1, x_2 \in P$ then

$$x = x_1 * x_2$$

$$f(x) = f(x_1) * f(x_2)$$

$$f(x_1) \mid f(x).$$

ii. For all $x \in P \setminus \{e\}$, $e = f(x) f^{-1}(x)$,

$$f^{-1}(x) = \frac{e}{f(x)}.$$

Lemma 12. Let $f:P\to P$ be a poloid mapping on poloid $(P,*)$. $(f(P),*)$ is a poloid, where $f(P)=\{f(x)|x\in P\}$.

Proof. Let $(P,*)$ be a poloid. For any $a\in P$ there exist $b,c,d\in P$ that $a=b*c=c*d$, where $b\neq d$. The property (P4) of $(f(P),*)$ is given by

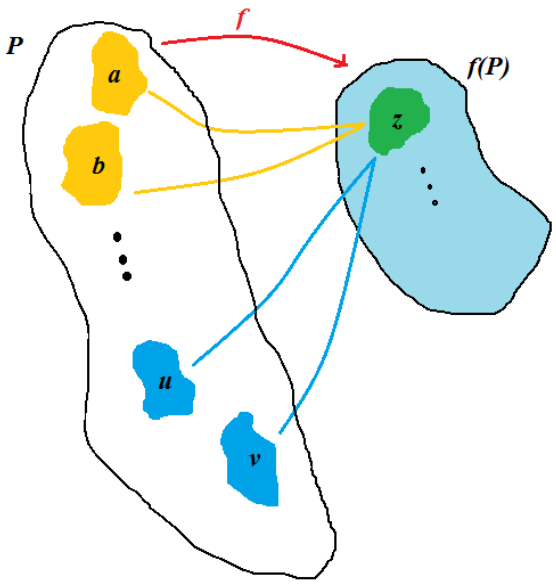
$$f(a)=f(b*c)=f(b)*f(c),$$

$$=f(c*d)=f(c)*f(d)$$

$$f(a)=f(b)*f(c)=f(c)*f(d), \text{ where } f(b)\neq f(d).$$

$(f(P),*)$ is a poloid.

For any $z\in f(P)$ there exist $a,b,\cdots,u,v\in P$ that $z=f(a)*f(b)=\cdots=f(u)*f(v)$, where $f(a)\neq f(u),\cdots,f(b)\neq f(v)$.



Example 13. If $M_n(\square)$ and the mapping f on poloid $(M_n(\square),.)$ is defined as

$$f:M_n(\square)\to M_n(\square)$$

$$X\to f(X)=X^k, \text{ where } k\in\square^+$$

For $A,B\in M_n(\square)$

Then

$$f(AB)=(AB)^k=A^kB^k=f(A)f(B),\quad f(A)\mid f(AB)\ .$$

The mapping f is the poloid mapping.

Example 14. If $M_n(\square)$ and the mapping g on poloid $(M_n(\square),.)$ is defined as

$$g:M_n(\square)\to M_n(\square)$$

$$X\to g(X)=e^X$$

For any $A,B\in M_n(\square)$

Then

$$g(AB)=e^{AB}=(e^A)^B,$$

$$g(AB)\neq g(A)g(B)\ .$$

The mapping g is not the poloid mapping.

Discussion and Conclusion

There are very studies about dividing matrices. The explanation was given about non-divisible matrices with this study. In addition, the study provided a new perspective for elements not associated with division. The transformation that relates to a poloid is given for the first time. New gains are expected for future transformations.

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