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On sequences arising from the action of modular group

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Abstract: This paper examines the sequences produced by the natural action of specific elements of the modular group on extended rational numbers. The polynomial sequences $P_r(c)$ and $Q_r(c)$ are derived from the orbit of the point at infinity under a specific modular transformation. These sequences satisfy linear recurrence relations, which are analyzed using generating functions. The polynomials encode k-Fibonacci numbers, and studying their behavior modulo a fixed integer m reveals notable arithmetic and combinatorial properties. We also explore the connection between these modular actions and Farey graphs, illustrating the hyperbolic transformations as nested geodesic paths in the upper half-plane.

Key words: Modular group action, recurrence relations, Fibonacci sequences

1. Introduction

The modular group and its subgroups play a central role in various branches of mathematics, including number theory, geometry, and combinatorics. The action of modular transformations on extended rational numbers, that is $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, encodes rich algebraic and combinatorial data (see, e.g., [9, 10, 16, 17]). This action has also been studied not only on $\hat{\mathbb{Q}}$ but also on real quadratic fields, yielding remarkable results [4, 5]. The study in [10] analyzes the modular group orbits using Farey graphs, which inspires part of our approach. Recent studies have established connections between the orbits of modular transformations and well-known integer sequences such as the Fibonacci, Lucas, and Pascal numbers (see, e.g., [1, 8, 13, 15]). The significance of continued fractions for the modular group arises from the fact that the action of Γ can be used to generate and represent such expansions. In particular, the orbit of ∞ under the action of the matrices in Γ traces the steps of continued fraction expansion and can be represented by paths in the Farey graphs [6]. Recent studies also show how matrix methods can effectively generate and analyze recurrence relations for special sequences of numbers. For example, in [2] new matrices are given that generate the Pell and Pell-Lucas numbers. In addition, a connection is established between suborbital graphs and integer solutions of the Pell equation [12].

This work builds on work [15] of the authors Şanlı and Köroğlu, in which they introduce the parametric matrices of the modular group.

$$T_c := \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

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Moreover, if one considers that every point on the orbit $\{T_c^r(\infty) : r \in \mathbb{N}\}$ can be expressed in the form $\frac{P_r(c)}{Q_r(c)}$, where $P_r(c), Q_r(c) \in \mathbb{Z}[c]$, the ring of polynomials in c with integer coefficients, and are given explicitly. Moreover, it is shown that the sum of the coefficients of these polynomials corresponds to the classical Fibonacci sequence.

In this study, we continue exploring this topic by looking at patterns and generating functions related to polynomial sequences $P_r(c)$ and $Q_r(c)$. We use the action of T_c to find the recurrence relations for both sequences and determine their corresponding generating functions. These results show a connection between modular transformations and generalized Fibonacci sequences. We also found an elegant formula for connecting continued fractions.

The novelty of this work lies in revealing a connection between the orbit structure of specific modular transformations and k-Fibonacci sequences, along with their continued fraction expansions.

2. Preliminaries

This section provides the necessary background on the modular group action, continued fractions, and generating functions required for our main results. The discussion includes the action of the modular group on extended rationals, continued fractions, recurrence relations, and generating functions. These notions will be used throughout the paper in connection with polynomial sequences arising from iterated modular transformations.

2.1. The modular group and its action on $\hat{\mathbb{Q}}$

The modular group $\Gamma := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ consists of equivalence classes of 2×2 integer matrices with determinant 1, under identification of a matrix with its negative. This group acts on $\hat{\mathbb{Q}}$ via the Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \mapsto \frac{ax+by}{cx+dy}, \quad \gcd(x,y) = 1.$$

In particular, ∞ is represented by $\frac{1}{0}$, and its image under γ is simply a/c (with $c \neq 0$). This action is well-defined and transitive. The following proposition emphasizes the meaning of the transformation T_c, which is the basis of this work.

Proposition 2.1 *For all $c \in \mathbb{Z}$, the matrix T_c can be written as a commutator in the modular group:*

$$T_c = \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix} = [s, t^c] = st^c s^{-1} t^{-c},$$

where $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the generators of Γ.

2.2. On numbers and sequences

Definition 2.2 *A finite simple continued fraction expansion of $\alpha \in \mathbb{Q}$ is an expression of the form:*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = [a_0; a_1, a_2, \ldots, a_n],$$

where a_i are positive integers for $1 \leq i \leq n$ and a_0 is an integer. Every rational number has a finite simple continued fraction expansion [11].

3. Main calculations and results

In this study, we focus on a specific family of elements in Γ parametrized by an integer $c \in \mathbb{Z}$, defined by

$$T_c := \begin{pmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{pmatrix}.$$

It can be verified that $\det(T_c) = 1$, so $T_c \in \mathrm{SL}_2(\mathbb{Z})$ and thus defines a class in Γ.

We consider the forward orbit of the point at infinity under the iterated action of T_c, namely,

$$\mathcal{O}_c := \{T_c^r(\infty) : r \in \mathbb{N}\}.$$

Since ∞ corresponds to the projective point $\frac{1}{0}$, iterations $T_c^r(\infty)$ are rational numbers that can be expressed as

$$T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)},$$

where $P_r(c), Q_r(c) \in \mathbb{Z}[c]$ are polynomials of c. These polynomials satisfy linear recurrence relations and connect with generalized Fibonacci sequences and continued fractions. For example, in [15] it is shown that the sums of the coefficients of $P_r(1)$ and $Q_r(1)$ are Fibonacci numbers.

This orbit construction is the starting point for investigating various algebraic and number-theoretic properties, which will be examined in the following sections.

3.1. Former related results

Theorem 3.1 *[Polynomial expressions for $T_c^r(\infty)$] The polynomials $P_r(c)$ and $Q_r(c) \in \mathbb{Z}[c]$ satisfy the following explicit combinatorial formulas:*

$$P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}, \tag{3.1}$$

$$Q_r(c) = \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+2}. \tag{3.2}$$

These polynomials describe the iterated action of T_c on the point at infinity in the extended rational line $\widehat{\mathbb{Q}}$. The proof of Theorem 3.1 was established in [15]. Furthermore, in the same work, the following identities given by Proposition 3.2 were proved that relate $P_r(c)$ and $Q_r(c)$ to the Fibonacci sequence $\{F_r\}$ such that $F_0 = 0$, $F_1 = 1$, and $F_r = F_{r-1} + F_{r-2}$ for $r \geq 2$:

Proposition 3.2 *i. The sum of the coefficients of $P_r(c)$ is equal to the Fibonacci number F_{2r+2}, that is, $P_r(1) = F_{2r+2}$.*

ii. The sum of the coefficients of $Q_r(c)$ is equal to F_{2r}, that is, $Q_r(1) = F_{2r}$.

iii. $P_r(-1) = F_{2r-1}$.

iv. The sum of the coefficients of the odd-degree terms of $P_r(c)$ *is equal to* F_{2r}.

v. The sum of the coefficients $P_r(c)$ *with even degree is* F_{2r+1}.

3.2. Connection with k-Fibonacci numbers

In this section, we prove the following identities (3.3) and (3.4) stated in [15] without proof:

$$P_r(c) = F_{c,2r} + F_{c,2r+1}, \tag{3.3}$$

$$Q_r(c) = c \cdot F_{c,2r}. \tag{3.4}$$

Here, $\{F_{k,n}\}$ denotes the k-Fibonacci sequence defined by:

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2.$$

The explicit formula for $F_{k,n}$ is given by [7]:

$$F_{k,n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} k^{n-2i}, \quad n \geq 1.$$

Proof of identities (3.3) and (3.4):

Using the formula above with $k = c$, we have:

$$\begin{aligned} F_{c,2r} + F_{c,2r+1} &= \sum_{i=0}^{r-1} \binom{2r-i-1}{i} c^{2r-2i-1} + \sum_{i=0}^{r} \binom{2r-i}{i} c^{2r-2i} \\ &= \sum_{i=1}^{r} \binom{2r-i}{i-1} c^{2r-2i+1} + \sum_{i=0}^{r} \binom{2r-i}{i} c^{2r-2i} \\ &= P_r(c). \end{aligned}$$

Similarly,

$$\begin{aligned} c \cdot F_{c,2r} &= c \cdot \sum_{i=0}^{r-1} \binom{2r-i-1}{i} c^{2r-2i-1} = \sum_{i=1}^{r} \binom{2r-i}{i-1} c^{2r-2i+2} \\ &= Q_r(c). \end{aligned}$$

Hence, (3.3) and (3.4) are valid. Hence, we have proved the following proposition.

Proposition 3.3 *The following equality is true for* $c \geq 1$ *and* $r \geq 1$,

$$T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)} = \frac{F_{c,2r} + F_{c,2r+1}}{c \cdot F_{c,2r}}.$$

3.3. Sequential and Farey graphical properties

In this subsection, we investigate the ordering and asymptotic behavior of the orbit $\{T_c^r(\infty)\}$ for different values of c and r. The values in the orbits for different c become widely separated, as is seen from the following proposition.

Proposition 3.4 *For all integers $c \geq 1$ and $r \geq 1$, the strict inequality*

$$T_{c+1}(\infty) < T_c^r(\infty)$$

holds.

Proof From the matrix definition, we know that

$$T_{c+1}(\infty) = \frac{(c+1)^2 + (c+1) + 1}{(c+1)^2} = 1 + \frac{c+2}{(c+1)^2}.$$

It is clear that

$$\frac{c+2}{(c+1)^2} < \frac{1}{c} \quad \text{for all } c \geq 1;$$

hence,

$$T_{c+1}(\infty) < 1 + \frac{1}{c}. \tag{3.5}$$

On the other hand, by Proposition 3.3, we have

$$T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)} = \frac{F_{c,2r} + F_{c,2r+1}}{c \cdot F_{c,2r}} = \frac{1}{c} + \frac{1}{c} \cdot \frac{F_{c,2r+1}}{F_{c,2r}}. \tag{3.6}$$

Since the k-Fibonacci sequence $F_{c,n}$ is strictly increasing for all $c \geq 1$, we have $\frac{F_{c,2r-1}}{F_{c,2r}} < 1$. Hence, we obtain, by the definition of k-Fibonacci sequence, that

$$\frac{F_{c,2r+1}}{F_{c,2r}} = \frac{c \cdot F_{c,2r} + F_{c,2r-1}}{F_{c,2r}} = c + \frac{F_{c,2r-1}}{F_{c,2r}} > c. \tag{3.7}$$

From the inequalities (3.6) and (3.7), we obtain

$$T_c^r(\infty) = \frac{1}{c} + \frac{1}{c} \cdot \frac{F_{c,2r+1}}{F_{c,2r}} > \frac{1}{c} + \frac{1}{c} \cdot c = \frac{1}{c} + 1, \tag{3.8}$$

Combining inequalities (3.5) and (3.8), we have

$$T_{c+1}(\infty) < 1 + \frac{1}{c} < \ T_c^r(\infty),$$

which completes the proof. □

Now, we consider the difference by fixing $r = 1$,

$$\delta(c) := T_c(\infty) - T_{c+1}(\infty) = \frac{c^2 + 3c + 1}{c^2(c+1)^2}.$$

The following proposition follows directly.

Proposition 3.5 *The sequence* $\{\delta(c)\}_{c=1}^{\infty}$ *is monotonically decreasing and converges to* 0.

Proposition 3.6 *The sequence* $\{T_c^r(\infty)\}_{r=1}^{\infty}$ *is monotonically decreasing and converges to* $\alpha_c := \frac{c^2+2-c\sqrt{c^2+4}}{2}$, *for any positive integer* c, *where* α_c *is attracting fixed point of* T_c.

Proof Since the trace of T_c satisfies $\mathrm{tr}(T_c) > 2$ for all $c \geq 1$, the transformation is hyperbolic. By the classical theory of Möbius transformations (see [14, Chapter 9]), a hyperbolic transformation has two fixed points, one attracting and one repelling, and iterates such as $T_c^r(\infty)$ converge to the attracting one.

Solving the fixed point equation for T_c:

$$z = \frac{(c^2+c+1)z - c}{c^2 z + (1-c)},$$

we obtain the attracting root as

$$z = \alpha_c = \frac{c^2+2-c\sqrt{c^2+4}}{2},$$

which is the limit of $T_c^r(\infty)$ as $r \to \infty$. The monotonicity is clear. □

Remark 3.7 *It is worth noting that the difference between consecutive terms of the orbit is given by the determinant formula:*

$$\left|T_c^r(\infty) - T_c^{r+1}(\infty)\right| = \left|\frac{P_r(c)}{Q_r(c)} - \frac{P_{r+1}(c)}{Q_{r+1}(c)}\right| = \frac{1}{Q_r(c)Q_{r+1}(c)}.$$

Since $Q_r(c)$ *takes integer values for all* r *and* c, *this difference is always a reciprocal integer (a unit fraction). Moreover, as* $Q_r(c)$ *grows geometrically with respect to* r, *this difference tends to 0 very rapidly. This observation provides a quantitative explanation for the fast convergence of the sequence* $\{T_c^r(\infty)\}$ *to its limit* α_c.

Now, we illustrate the Farey graphs obtained from this action. First, we recall some basics about Farey sequences and graphs. The Farey graph is a classical object in number theory and hyperbolic geometry. It is defined as follows:

Vertices and edges

The set of vertices of the Farey graph consists of all rational numbers (including infinity) in the lowest terms:

$$\mathbb{Q} \cup \{\infty\} = \left\{ \frac{a}{b} \in \mathbb{Q} \,\middle|\, \gcd(a,b) = 1,\, b > 0 \right\} \cup \{\infty\}.$$

Two vertices $\frac{a}{b}$ and $\frac{c}{d}$ are connected by an edge if and only if

$$|ad - bc| = 1.$$

This condition guaranties that the two fractions are adjacent in some Farey sequence.

Geometric embedding

The Farey graph can be embedded in the upper half-plane model of hyperbolic geometry. Each rational number is represented as a boundary point on the real line, and each edge is drawn as a geodesic arc (semicircle orthogonal to the real axis) connecting two of these points illustrated in Figure. For each c, the first few iterations of the transformation T_c applied to ∞ are illustrated using semicircular arcs connecting consecutive rational points on the real line. The black arcs represent the initial step $T_c(\infty)$, and the blue arcs show the subsequent steps $T_c^2(\infty)$, $T_c^3(\infty)$, etc. This figure demonstrates how the orbits evolve geometrically depending on the parameter c, and highlights the nested behavior of their trajectories in the context of hyperbolic geometry.

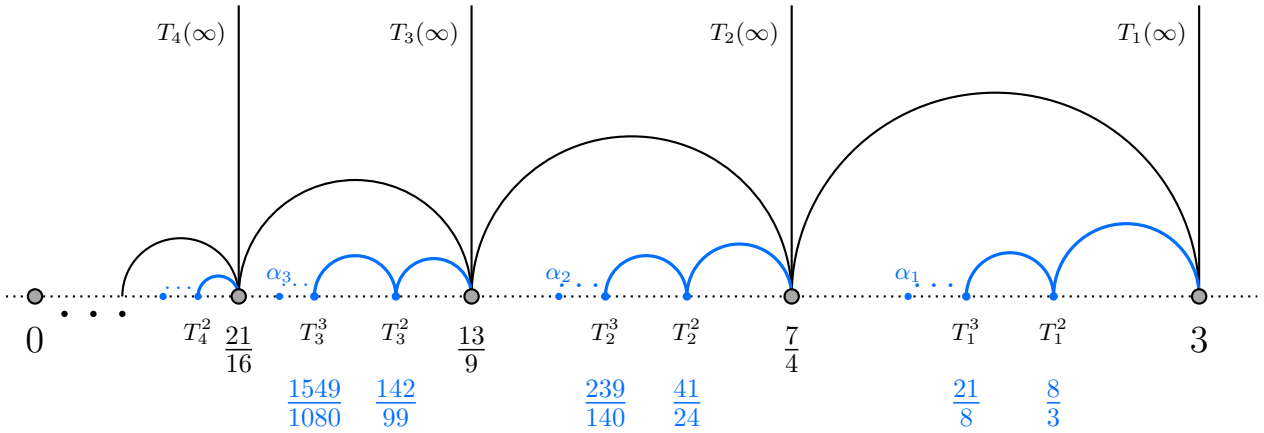


Figure. The Farey Graph of T_c in $\mathbb{H}$.

3.4. Recurrence Relations for the Polynomial Sequences $P_r(c)$ and $Q_r(c)$

In this section, we derive the linear recurrence relations satisfied by the polynomials $P_r(c)$ and $Q_r(c)$.

Proposition 3.8 *Polynomials $P_r(c)$ and $Q_r(c)$ satisfy the recurrence relations:*

$$P_{r+1}(c) = (c^2 + c + 1)P_r(c) - cQ_r(c),$$
$$Q_{r+1}(c) = c^2 P_r(c) + (1 - c)Q_r(c),$$

with initial conditions

$$P_0(c) = 1, \quad Q_0(c) = 0.$$

Proof The action of T_c on a fraction $\frac{P_r(c)}{Q_r(c)}$ is given by

$$T_c\left(\frac{P_r(c)}{Q_r(c)}\right) = \frac{(c^2 + c + 1)P_r(c) - cQ_r(c)}{c^2 P_r(c) + (1 - c)Q_r(c)}.$$

Thus, iteratively, if $T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)}$, then

$$T_c^{r+1}(\infty) = T_c\left(T_c^r(\infty)\right) = \frac{P_{r+1}(c)}{Q_{r+1}(c)},$$

where

$$P_{r+1}(c) = (c^2 + c + 1)P_r(c) - cQ_r(c),$$
$$Q_{r+1}(c) = c^2 P_r(c) + (1 - c)Q_r(c),$$

as claimed. □

Corollary 3.9 *The eigenvalues of T_c are the roots of the quadratic polynomial*

$$\lambda^2 - (c^2 + 2)\lambda + 1 = 0,$$

that is

$$\lambda = \frac{c^2 + 2 \pm c\sqrt{c^2 + 4}}{2}.$$

3.5. Generating functions associated with the polynomial sequences

In this section, we establish the generating functions for the sequences of polynomials $P_r(c)$ and $Q_r(c)$, which arise from the iterated action of the matrix T_c on ∞.

Definition 3.10 *If $A = \{A_n\}_{n=b}^{\infty} = \{A_b, A_{b+1}, \dots\}$ is a sequence of real numbers, then*

$$G_A(x) := \sum_{n=0}^{\infty} A_n x^n,$$

is called the generating function for the sequence $\{A_n\}$ [3].

Proposition 3.11 *Polynomial sequences $\{P_r(c)\}$ and $\{Q_r(c)\}$ satisfy the explicit formulas*

$$P_r(c) = \left(\frac{c^2 + c + 1 - \beta}{\alpha - \beta}\right)\alpha^r + \left(\frac{\alpha - (c^2 + c + 1)}{\alpha - \beta}\right)\beta^r,$$

$$Q_r(c) = \frac{c}{\alpha - \beta}\left(\alpha^r - \beta^r\right),$$

where

$$\alpha = \frac{c^2 + 2 + \sqrt{c^4 + 4c^2}}{2}, \qquad \beta = \frac{c^2 + 2 - \sqrt{c^4 + 4c^2}}{2}.$$

Proof Recurrence relations

$$P_{r+1}(c) = (c^2 + 2)P_r(c) - P_{r-1}(c), \quad Q_{r+1}(c) = (c^2 + 2)Q_r(c) - Q_{r-1}(c)$$

are second-order linear recurrences with constant coefficients. The characteristic equation

$$x^2 - (c^2 + 2)x + 1 = 0$$

has roots α and β as given.

Thus, the general solutions are of the form

$$P_r(c) = A\alpha^r + B\beta^r, \qquad Q_r(c) = C\alpha^r + D\beta^r.$$

Applying the initial conditions

$$P_0 = 1, \quad P_1 = c^2 + c + 1, \quad Q_0 = 0, \quad Q_1 = c,$$

we obtain:

$$A + B = 1, \quad A\alpha + B\beta = c^2 + c + 1,$$

$$C + D = 0, \quad C(\alpha - \beta) = c.$$

Solving these systems gives the explicit formulas for $P_r(c)$ and $Q_r(c)$ as stated. □

Proposition 3.12 *The generating functions of the sequences* $\{P_r(c)\}_{r\geq 0}$ *and* $\{Q_r(c)\}_{r\geq 0}$ *are given by*

$$\mathcal{P}_c(x) = \sum_{r=0}^{\infty} P_r(c)x^r = \frac{1 + (c^2 + c)x}{1 - (c^2 + 2)x + x^2},$$

$$\mathcal{Q}_c(x) = \sum_{r=0}^{\infty} Q_r(c)x^r = \frac{cx}{1 - (c^2 + 2)x + x^2}.$$

Proof Both sequences $\{P_r(c)\}$ and $\{Q_r(c)\}$ satisfy the second-order linear recurrence

$$f_{r+1} = (c^2 + 2)f_r - f_{r-1},$$

with initial conditions

$$P_0 = 1, \quad P_1 = c^2 + c + 1, \qquad Q_0 = 0, \quad Q_1 = c.$$

Let $\mathcal{P}_c(x) = \sum_{r=0}^{\infty} P_r(c)x^r$ be the generating function of $\{P_r(c)\}$. Then, multiplying the recurrence by x^{r+1} and summing over r, we obtain the following:

$$\sum_{r=0}^{\infty} P_{r+1}(c)x^{r+1} = (c^2 + 2)\sum_{r=0}^{\infty} P_r(c)x^{r+1} - \sum_{r=0}^{\infty} P_{r-1}(c)x^{r+1}.$$

Shifting indices and rearranging terms yields the following

$$\mathcal{P}_c(x) - P_0 - P_1x = (c^2 + 2)x\mathcal{P}_c(x) - x^2\mathcal{P}_c(x).$$

Solving for $\mathcal{P}_c(x)$, we find

$$\mathcal{P}_c(x) = \frac{1 + (c^2 + c)x}{1 - (c^2 + 2)x + x^2}.$$

A similar computation applies to $\mathcal{Q}_c(x)$, leading to the stated expression. □

Remark 3.13 *The denominators of* $\mathcal{P}_c(x)$ *and* $\mathcal{Q}_c(x)$ *coincide and reflect the characteristic equation*

$$x^2 - (c^2 + 2)x + 1 = 0$$

associated with the recurrence relation of the sequences.

Example 1: We illustrate the recurrence relations for $P_r(c)$ and $Q_r(c)$ in the case $c=1$.

Using the initial conditions,

$$P_0(1)=1, \quad P_1(1)=3, \qquad Q_0(1)=0, \quad Q_1(1)=1,$$

we get the recurrence

$$f_{r+1}=3f_r-f_{r-1},$$

and we can compute successive terms of the sequences.

The first values are summarized in Table 1.

Table 1. Values of $P_r(1)$ and $Q_r(1)$ for small r.

r	$P_r(1)$	$Q_r(1)$
0	1	0
1	3	1
2	8	3
3	21	8
4	55	21
5	144	55

Remark 3.14 *As observed in Table 1, both sequences $P_r(1)$ and $Q_r(1)$ exhibit rapid growth that precisely follows Fibonacci numbers.*

3.6. Connection with continued fractions

The relation of the rational numbers produced by the transformation T_c with continued fractions is given by the following proposition.

Proposition 3.15 *For all $r\in\mathbb{N}$ and $c\geq 2$, the rational function $\frac{P_r(c)}{Q_r(c)}$ has the continued fraction expansion:*

$$\frac{P_r(c)}{Q_r(c)}=\left[1,c-1,\underbrace{c,c,\ldots,c}_{2r-2\ times},c+1\right].$$

Proof Let us prove this by induction on r.

Base Case: For $r=1$, we have

$$P_1(c)=c^2+c+1, \quad Q_1(c)=c^2 \Rightarrow \frac{P_1(c)}{Q_1(c)}=\frac{c^2+c+1}{c^2}=1+\frac{c+1}{c^2}.$$

This can be written as the continued fraction $[1,c-1,c+1]$.

Inductive Step: Assume the statement holds for r. That is,

$$\frac{P_r(c)}{Q_r(c)}=[1,c-1,\underbrace{c,c,\ldots,c}_{2r-2\ \text{times}},c+1].$$

From Theorem 3.1, we know:

$$\frac{P_{r+1}(c)}{Q_{r+1}(c)}=T_c\left(\frac{P_r(c)}{Q_r(c)}\right), \quad \text{where } T_c(z)=\frac{(c^2+c+1)z-c}{c^2z+(1-c)}.$$

Applying T_c adds two more entries c to the continued expansion of the fraction. Hence,

$$\frac{P_{r+1}(c)}{Q_{r+1}(c)} = [1, c-1, \underbrace{c, \ldots, c}_{2r \text{ times}}, c+1],$$

which completes the inductive proof. □

Remark 3.16 *In case $c = 1$, the Proposition 3.15 is not valid. In this case, considering Proposition 3.2, we can write $\frac{P_r(1)}{Q_r(1)} = \frac{F_{2r+2}}{F_{2r}}$, where F_n denotes the classical Fibonacci numbers.*

Example 2: Let us illustrate the result with $c = 2$ and $r = 2$. From Theorem 3.1, we have:

$$P_2(2) = \sum_{s=0}^{2} \binom{4-s}{s} 2^{4-2s} + \sum_{s=1}^{2} \binom{4-s}{s-1} 2^{5-2s} = 41.$$

$$Q_2(2) = \sum_{s=1}^{2} \binom{4-s}{s-1} 2^{6-2s} = 24.$$

Therefore,

$$\frac{P_2(2)}{Q_2(2)} = \frac{41}{24}.$$

On the other hand, using the continued fraction formula, it is easily checked:

$$[1,1,2,2,3] = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3}}}} = \frac{41}{24}.$$

3.7. Modular properties of the polynomials $P_r(c)$ and $Q_r(c)$

In this section, we examine the modular behavior of polynomials $P_r(c)$ and $Q_r(c)$. Since these polynomials encode k-Fibonacci numbers in their structure, understanding their behavior modulo a fixed integer m reveals interesting arithmetic and periodic properties. The periodicity of the Fibonacci sequence modulo a positive integer m, known as *Pisano period*, is the smallest positive integer $\pi(m)$ such that $F_{n+\pi(m)} \equiv F_n \mod m$ for all $n \in \mathbb{N}$ (see [18]).

Consider the reduction of these polynomials modulo m, for example:

$$P_r(c) \bmod m \quad \text{and} \quad Q_r(c) \bmod m.$$

Example 3: Reducing the coefficients of $P_r(c)$ and $Q_r(c)$ modulo 2 leads to binary sequences, which are easier to analyze for patterns. In particular, we observe that

$$P_r(1) = F_{2r+2} \equiv F_{2r+2} \bmod 2.$$

Given the periodicity of the Fibonacci sequence modulo 2 (with period 3), we obtain:

$$P_r(1) \equiv \{1, 1, 0, 1, 1, 0, \dots\} \mod 2.$$

Example 4: Fixing $c = 1$ and computing $P_r(1) \bmod m$ for various r, we observe that the sequence is a small periodic modulo primes. This suggests that $P_r(1) \mod m$ might inherit the Pisano periodicity of the Fibonacci sequence.

We aim to generalize such results and examine whether similar modular periodicity or congruences can be established for $P_r(c)$ and $Q_r(c)$ under various values of c and m.

Proposition 3.17 *Let $P_r(c)$ be the polynomial defined in Theorem 3.1. Then the sequence $\{P_r(1)\}_{r\geq 0} \mod 2$ is periodic with period 3, and it satisfies*

$$P_r(1) \equiv \begin{cases} 1 \mod 2, & \text{if } r \equiv 0 \text{ or } 1 \mod 3, \\ 0 \mod 2, & \text{if } r \equiv 2 \mod 3. \end{cases}$$

Proof By Theorem 3.1, we have the identity $P_r(1) = F_{2r+2}$, where F_n denotes the n-th Fibonacci number. It is a well-known property of the Fibonacci sequence that $F_n \mod 2$ is periodic with period 3, and the sequence is given by:

$$F_n \mod 2 = \{0, 1, 1, 0, 1, 1, 0, \dots\}.$$

Therefore, the values of $F_{2r+2} \mod 2$ will also repeat with a period of 3. Observing the values of $F_{2r+2} \mod 2$, we find:

$$F_2 = 1 \equiv 1 \mod 2,$$

$$F_4 = 3 \equiv 1 \mod 2,$$

$$F_6 = 8 \equiv 0 \mod 2,$$

$$F_8 = 21 \equiv 1 \mod 2, \quad \text{etc.}$$

So the sequence $\{P_r(1) \mod 2\}$ is:

$$1,\ 1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ 0,\ \dots,$$

which has a repeating pattern $(1,\ 1,\ 0)$. This implies the result. □

4. Conclusion and future work

In this paper, we investigate the rational orbits generated by the iterated action of the matrix

$$T_c = \begin{pmatrix} c^2 + c + 1 & -c \\ c^2 & 1 - c \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and examined their connections with k-Fibonacci sequences, generating functions, recurrence relations, and continued fractions. Furthermore, we present a continued fraction expansion of the form

$$\left[1, c-1, \underbrace{c, c, \dots, c}_{2r-2 \text{ times}}, c+1\right],$$

which describes the orbit under $T_c^r(\infty)$ when $c \geq 2$. In the special case $c = 1$, the orbit simplifies and yields ratios of classical Fibonacci numbers. We also analyzed the geometric behavior of these orbits in the upper half-plane and visualized their edges as hyperbolic geodesics, revealing both algebraic and geometric structure. In particular, we showed that the orbit limit points correspond to the attracting fixed points of hyperbolic transformations, and we derived their explicit forms. In the final part of the study, we explored the modular behavior of $P_r(c)$ and $Q_r(c)$.

Future directions

Future research could investigate the spectral properties and trace fields associated with the subgroups generated by T_c, as well as the interaction between their orbits and modular symbols. In addition, exploring the orbit structure induced by various congruence subgroups, such as $\Gamma_0(N)$, $\Gamma_1(N)$, or $\Gamma(N)$, can reveal deeper algebraic and geometric properties.

Author contributions

TK contributed to conceptualization, investigation, and data curation. BOG led the methodology and supervision. Both authors contributed equally to the original draft preparation and to the review and editing of the manuscript.

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