Hindawi
Advances in Fuzzy Systems
Volume 2019, Article ID 3693926, 7 pages
https://doi.org/10.1155/2019/3693926

Research Article

On Fuzzy Ordered Hyperideals in Ordered Semihyperrings

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Received 28 May 2018; Accepted 12 December 2018; Published 3 February 2019

Academic Editor: Antonin Dvorák

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In this paper, we introduce the concept of fuzzy ordered hyperideals of ordered semihyperrings, which is a generalization of the concept of fuzzy hyperideals of semihyperrings to ordered semihyperring theory, and we investigate its related properties. We show that every fuzzy ordered quasi-hyperideal is a fuzzy ordered bi-hyperideal, and, in a regular ordered semihyperring, fuzzy ordered quasi-hyperideal and fuzzy ordered bi-hyperideal coincide.

1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory which was initiated by Marty [1]. Since then many researchers have worked on algebraic hyperstructures and developed it [2, 3]. A short review of this theory appears in [4–8].

The notion of semiring was introduced by Vandiver [9] in 1934, which is a generalization of rings. Semirings are very useful for solving problems in graph theory, automata theory, coding theory, analysis of computer programs, and so on. We refer to [10] for the information we need concerning semiring theory. In [11–13], quasi-ideals of semirings are studied and some properties and related results are given. In [8], Vougiouklis generalized the notion of hyperring and named it as semihyperring, where both the addition and multiplication are hyperoperations. Semihyperrings are a generalization of Krasner hyperrings. Davvaz, in [14], studies the notion of semihyperring in a general form. Ameri and Hedayati define k-hyperideals in semihyperrings in [15]. In 2011, Heidari and Davvaz [16] studied a semihypergroup $(H, \circ)$ with a binary relation $\leq$, where $\leq$ is a partial order so that the monotony condition is satisfied. This structure is called an ordered semihypergroup. Properties of hyperideals in ordered semihypergroups are studied in [17]. Also, the properties of fuzzy hyperideals in an ordered semihypergroup are investigated in [18, 19]. Yaqoop and Gulistan [20] study the concept of ordered LA-semihypergroup. In [21], Davvaz and Omidi introduce the basic notions and properties of ordered semihyperrings and prove some results in this respect. In 2018, Omidi and Davvaz [22] studied on special kinds of hyperideals in ordered semihyperrings. Some properties of hyperideals in ordered Krasner hyperrings can be found in [23].

After the introduction of fuzzy sets by Zadeh [24], reconsideration of the concept of classical mathematics began. Because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroup is defined by Rosenfeld [25] and its structure is investigated. This subject has been studied further by many others [26, 27]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the world both on the theoretical point of view and for their many applications. There exists a rich bibliography: publications that appeared within 2015 can be found in "Fuzzy Algebraic Hyperstructures - An Introduction" by Davvaz and Cristea [28]. Recently, many researchers have considered fuzzification on many algebraic structures, for example, on semigroups, rings, semirings, near-rings, ordered semigroups, semihypergroups, ordered semihypergroups, and ordered hyperrings [29–34].

Inspired by the study on ordered semihyperrings, we study the concept of fuzzy ordered hyperideals, fuzzy ordered quasi-hyperideals, and fuzzy ordered bi-hyperideals of an ordered semihyperring and we present some examples in this respect. The rest of this paper is organized as follows.

In the second section, we review basic concepts regarding the ordered hyperstructures. The third section is dedicated to the fuzzy ordered hyperideals and some properties. In Section 4, we introduce the concept of fuzzy ordered quasi-hyperideals and fuzzy ordered bi-hyperideals of an ordered semihyperring and we present some examples. We give the main theorems which characterize the ordered hyperideals, ordered quasi-hyperideals, and ordered bi-hyperideals in terms of fuzzy ordered hyperideals, fuzzy ordered quasi-hyperideals, and fuzzy ordered bi-hyperideals, respectively.

2. Terminology and Basic Properties

In what follows, we summarize some basic notions and facts about semihypergroups, semihyperrings, and ordered semihyperrings.

Let H be a nonempty set and let $\mathscr{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathscr{P}^*(H)$ and the pair $(H, \circ)$ is called a *hypergroupoid*. For any $x \in H$ and $A, B \in \mathscr{P}^*(H)$, we denote

$$A \circ B = \bigcup \{x \circ y \mid x \in A, y \in B\},$$
$$x \circ A = \{x\} \circ A, \tag{1}$$
$$A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$ we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v \tag{2}$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if, for all $x \in H$, we have $x \circ H = H \circ x = H$.

Definition 1 (see [8, 21]). *A semihyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(i) $(R, +)$ is a commutative semihypergroup

(ii) $(R, \cdot)$ is a semihypergroup

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$

Let $(R, +, \cdot)$ be a semihyperring. If there exists an element $0 \in R$ such that $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, then 0 is called the *zero element* of R. Throughout this paper we consider a semihyperring $(R, +, \cdot)$ with zero element 0.

A semihyperring R is called *commutative* if $(R, \cdot)$ is a commutative semihypergroup.

Definition 2 (see [21]).

(i) A nonempty subset A of a semihyperring $(R, +, \cdot)$ is called a *subsemihyperring* of R if, for all $x, y \in A$, $x + y \subseteq A$ and $x \cdot y \subseteq A$.

(ii) A nonempty subset I of a semihyperring $(R, +, \cdot)$ is called a *hyperideal* of $(R, +, \cdot)$ if, for all $x, y \in I, r \in R$, $x + y \subseteq I$ and $r \cdot x \subseteq I$ and $x \cdot r \subseteq I$.

Definition 3 (see [21]). An *ordered semihyperring* $(R, +, \cdot, \leq)$ is a semihyperring equipped with a partial order relation $\leq$ such that for all $a, b, c \in R$ we have the following.

(i) $a \leq b$ implies $a + c \leq b + c$, meaning that, for any $x \in a + c$, there exists $y \in b + c$ such that $x \leq y$.

(ii) $a \leq b$ and $0 \leq c$ imply $a \cdot c \leq b \cdot c$, meaning that, for any $x \in a \cdot c$, there exists $y \in b \cdot c$ such that $x \leq y$. The case $c \cdot a \leq c \cdot b$ is defined similarly.

Note that the concept of ordered semihyperring is a generalization of the concept of ordered semiring.

Semihyperrings are viewed as ordered semihyperrings under the equality order relation [21]. Indeed, let $(R, +, \cdot)$ be a semihyperring. Define the order relation on R by $\leq := \{(a, b) \mid a = b\}$. Then $(R, +, \cdot, \leq)$ is an ordered semihyperring.

Let A be a nonempty subset of an ordered semihyperring R. Then the set $\{x \in R \mid x \leq a \text{ for some } a \in A\}$ is denoted by the notation $(A]$. For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. An ordered semihyperring $(A, +, \cdot, \leq)$ is an *ordered subsemihyperring* of $(R, +, \cdot, \leq)$ if A is a subsemihyperring of R and the order on A is the restriction of the order on R. Let $(R, +, \cdot)$ and $(S, \oplus, \odot)$ be semihyperrings. A mapping $\varphi : R \longrightarrow S$ is said to be *strong homomorphism* if $\varphi(x + y) \subseteq \varphi(x) \oplus \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \odot \varphi(y)$ for all $x, y \in R$. A *homomorphism* of ordered semihyperrings $\varphi : (R, +, \cdot, \leq_1) \longrightarrow (S, \oplus, \odot, \leq_2)$ is a semihyperring homomorphism such that, for all $a, b \in R$, $a \leq_1 b$ implies $\varphi(a) \leq_2 \varphi(b)$.

Definition 4 (see [21]). Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A nonempty subset I of R is called an *ordered hyperideal* of R if it satisfies the following conditions:

(i) $x + y \subseteq I$ for all $x, y \in I$

(ii) $r \cdot x \subseteq I$ and $x \cdot r \subseteq I$ for all $x \in I$ and $r \in R$

(iii) If $x \in I$ and $R \ni r \leq x$, then $r \in I$

It is clear that $\{0\}$ and R are ordered hyperideals of R.

Example 5. Let $R = \{0, a, b, c\}$ and let the hyperoperations "$\oplus$" and "$\odot$" on R be defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	{0, a, b}	{0, a, c}
b	b	{0, a, b}	{0, b}	{0, b, c}
c	c	{0, a, c}	{0, b, c}	{0, c}

$\odot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	{0, a}
c	0	0	{0, a}	{0, b}

(3)

Then, $(R, \oplus, \odot)$ is a semihyperring [35]. It is easy to see that $I = \{0, a\}$, $J = \{0, a, b\}$ are hyperideals of R. $K = \{0, a, c\}$ is not a hyperideal of R.

Definition 6 (see [21]). Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A nonempty subset A of R is called an *ordered bi-hyperideal* of R if it satisfies the following conditions:

(i) A is a subsemihyperring of R

(ii) $A \cdot R \cdot A \subseteq A$

(iii) If $x \in A$ and $R \ni r \leq x$, then $r \in A$

Definition 7 (see [21]). Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A nonempty subset Q of R is called an *ordered quasi-hyperideal* of R if it satisfies the following conditions:

(i) $Q + Q \subseteq Q$

(ii) $(Q \cdot R] \cap (R \cdot Q] \subseteq Q$

(iii) If $x \in Q$ and $R \ni r \leq x$, then $r \in Q$

Example 8. Consider the semihyperring defined in Example 5. Then $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (0,a), (0,b), (0,c)\}. \quad (4)$$

The covering relation is given by

$$\prec = \{(0,a), (0,b), (0,c)\}. \quad (5)$$

Now, it is easy to see that $A = \{0, b\}$ is an ordered bi-hyperideal of R but it is not an ordered quasi-hyperideal of R.

The concept of a fuzzy subset of a nonempty set first was introduced by Zadeh in 1965 [36]. Let X be a nonempty set. A *fuzzy subset* μ of X is a function $\mu : X \longrightarrow [0,1]$. Let μ and λ be two fuzzy subsets of X; we say that μ *is contained in* λ and we write $\mu \subseteq \lambda$, if $\mu(x) \leq \lambda(x)$ for all $x \in X$, and $\mu \cap \lambda$, $\mu \cup \lambda$ are defined by $(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$ and $(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$. The sets $\mu_t = \{x \in X \mid t \leq \mu(x)\}$ and $\mu_t^{>} = \{x \in X \mid t < \mu(x)\}$, $t \in [0,1]$, are called a *level subset* and *strong level subset* of μ, respectively.

3. On Fuzzy Ordered Hyperideals in Ordered Semihyperrings

Notice that the relationships between fuzzy sets and algebraic hyperstructures have been already considered by many researchers [18, 28, 30, 35–38]. Recently, ordered ideals in semirings and ordered ideals in Krasner hyperrings have been already considered by Gan and Jiang [39] and Davvaz and Loeranau-Fotea [30], respectively. So, it is interesting to study fuzzy ordered hyperideals of ordered semihyperrings.

Definition 9. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring and let μ be a fuzzy subset of R. μ is called a *fuzzy ordered hyperideal* of R if the following conditions hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y}\{\mu(z)\}$ for all $x, y \in R$

(ii) $\max\{\mu(x), \mu(y)\} \leq \inf_{z \in x \cdot y}\{\mu(z)\}$ for all $x, y \in R$

(iii) $x \leq y \Longrightarrow \mu(x) \geq \mu(y)$ for all $x, y \in R$

Example 10. Let $R = \{0, a, b, c\}$ and the hyperoperations "$\oplus$" and "$\odot$" on R be defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	b
b	b	a	{0, b}	{0, b, c}
c	c	a	{0, b, c}	{0, c}

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	{0, b}	0
b	0	0	0	0
c	0	{0, c}	0	0

(6)

Then, $(R, \oplus, \odot)$ is a semihyperring [21]. Now, the order relation $\leq$ on R is defined by

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (0,a), (0,b), (0,c), (b,a), (c,a)\}. \quad (7)$$

The covering relation is given by

$$\prec = \{(0,b), (0,c), (b,a), (c,a)\} \quad (8)$$

Then, $(R, \oplus, \odot, \leq)$ is an ordered semihyperring. Let $\mu : R \longrightarrow [0,1]$ be defined by

$$\mu(0) = 1, \quad \mu(a) = 0.1, \quad \mu(b) = 0.2, \quad \mu(c) = 0.3. \quad (9)$$

Then, μ is a fuzzy ordered hyperideal of R.

Lemma 11. *Any hyperideal of an ordered semihyperring $(R, +, \cdot, \leq)$ can be realized as a level subset of some ordered fuzzy hyperideals of R.*

Proof. Proof is similar to Lemma 4.2 in [30]. □

Notice that the characteristic function of a nonempty subset I of an ordered semihyperring R is a fuzzy ordered hyperideal of R if and only if I is an ordered hyperideal of R.

Theorem 12. *A fuzzy subset μ of an ordered semihyperring R is a fuzzy ordered hyperideal of R if and only if the set $\mu_t (\neq \emptyset)$ is an ordered hyperideal of R for all $t \in [0,1]$.*

Proof. Let μ be a fuzzy ordered hyperideal of R and $t \in [0,1]$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$, $\mu(y) \geq t$. Now we have

$$\inf_{z \in x+y}\{\mu(z)\} \geq \min\{\mu(x), \mu(y)\} \geq t. \quad (10)$$

Therefore, for every $z \in x+y$, we have $\mu(z) \geq t$; that is, $z \in \mu_t$, so $x + y \subseteq \mu_t$. Let $x \in \mu_t$ and $r \in R$. Then $\mu(x) \geq t$. So

$$t \leq \mu(x) \leq \max\{\mu(x), \mu(r)\} \leq \inf_{z \in r \cdot x}\{\mu(z)\} \tag{11}$$

Therefore, for every $z \in r \cdot x$, we have $\mu(z) \geq t$; that is, $z \in \mu_t$, so $r \cdot x \subseteq \mu_t$. Similarly, $x \cdot r \subseteq \mu_t$. Now, let $x \in \mu_t$ and $y \in R$ such that $y \leq x$. Then $\mu(x) \geq t$. Since $y \leq x$, it follows that $\mu(y) \geq \mu(x) \geq t$. This implies that $y \in \mu_t$. By Definition 4, μ_t is an ordered hyperideal of R.

Conversely, let μ be a fuzzy subset of an ordered semihyperring R such that $\mu_t(\neq \emptyset)$ is an ordered hyperideal of R for all $0 \leq t \leq 1$. Let $t_0 = \min\{\mu(x), \mu(y)\}$ for $x, y \in R$. Then obviously $x, y \in \mu_{t_0}$. Since every nonempty level set is an ordered hyperideal, $x + y \subseteq \mu_{t_0}$. Thus

$$\min\{\mu(x), \mu(y)\} = t_0 \leq \inf_{z \in x+y}\{\mu(z)\}. \tag{12}$$

Let $t_1 = \max\{\mu(x), \mu(y)\}$ for $x, y \in R$. Then obviously $x, y \in \mu_{t_1}$. Since every nonempty level set is an ordered hyperideal, $x \cdot y \subseteq \mu_{t_1}$. Then, we obtain $\mu(x \cdot y) \geq t_1$. Thus

$$t_1 = \max\{\mu(x), \mu(y)\} \leq \mu(x \cdot y) \tag{13}$$

for all $x, y \in R$; that is, $\max\{\mu(x), \mu(y)\} \leq \inf_{z \in x \cdot y}\{\mu(z)\}$.

Finally, let $x, y \in R$ such that $x \leq y$. Let $t_2 = \mu(y)$; then $y \in \mu_{t_2}$. Since μ_{t_2} is an ordered hyperideal of R, $x \in \mu_{t_2}$. Thus $\mu(x) \geq t_2 = \mu(y)$. This completes the proof. □

Corollary 13. *Let μ be a fuzzy set with the upper bound t_0 of an ordered semihyperring R. Then the following conditions are equivalent:*

(i) *μ is a fuzzy ordered hyperideal of R*

(ii) *Each level subset μ_t, for $t \in [0, t_0]$, is an ordered hyperideal of R*

(iii) *Each strong level subset $\mu_t^>$, for $t \in [0, t_0]$, is an ordered hyperideal of R*

(iv) *Each level subset μ_t, for $t \in \mathrm{Im}(\mu)$, is an ordered hyperideal of R, where $\mathrm{Im}(\mu)$ denotes the image of μ*

(v) *Each strong level subset $\mu_t^>$, for $t \in \mathrm{Im}(\mu) - \{t_0\}$, is an ordered hyperideal of R*

(vi) *Each nonempty level subset of μ is an ordered hyperideal of R*

(vii) *Each nonempty strong level subset of $\mu_t^>$ is an ordered hyperideal of R*

Let φ be a mapping from an ordered semihyperring R_1 to an ordered semihyperring R_2. Let μ be a fuzzy subset of R_1 and let λ be a fuzzy subset of R_2. Then the inverse image *$\varphi^{-1}(\lambda)$ of λ is a fuzzy subset of R_1 defined by $\varphi^{-1}(\lambda)(x) = \lambda(\varphi(x))$ for all $x \in R_1$. The image $\varphi(\mu)$ of μ is the fuzzy subset of R_2 defined by*

$$\varphi(\mu)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)}\{\mu(x)\} & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

for all $y \in R_2$.

Lemma 14. *Let R_1 and R_2 be two ordered semihyperrings and let $\varphi : R_1 \longrightarrow R_2$ be a strong homomorphism.*

(i) *If μ is a fuzzy ordered hyperideal of R_1, then $\varphi(\mu)$ is a fuzzy ordered hyperideal of R_2.*

(ii) *If λ is a fuzzy ordered hyperideal of R_2, then $\varphi^{-1}(\lambda)$ is a fuzzy ordered hyperideal of R_1.*

Proof. It is straightforward. □

4. Fuzzy Ordered Bi-Hyperideals and Fuzzy Ordered Quasi-Hyperideals of Ordered Semihyperrings

In this section, we define the concepts of fuzzy ordered bi-hyperideal and fuzzy ordered quasi-hyperideal in ordered semihyperrings and give relationships between them.

Let $(R, +, \cdot, \leq)$ be an ordered semihyperring and $a \in R$. We denote

$$A_a = \{(b, c) \in R \times R \mid a \leq b \cdot c\} \tag{15}$$

[30]. For fuzzy subsets μ and λ of a semihyperring R, we define the fuzzy subset $\mu \circ \lambda$ of R by letting $a \in R$;

$$(\mu \circ \lambda)(a) = \begin{cases} \sup_{(b,c) \in A_a}\{\min\{\mu(b), \lambda(c)\}\} & \text{if } A_a \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

We denote the constant function $1 : R \longrightarrow [0, 1]$ defined by $1(a) = 1$ for all $a \in R$ [30].

Definition 15. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring and let μ be a fuzzy subset of R. Then, μ is called a *fuzzy ordered bi-hyperideal* of R if the following conditions hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y}\{\mu(z)\}$ and $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x \cdot y}\{\mu(z)\}$ for all $x, y \in R$

(ii) $\min\{\mu(x), \mu(z)\} \leq \inf_{w \in x \cdot y \cdot z}\{\mu(w)\}$ for all $x, y, z \in R$

(iii) $x \leq y \Longrightarrow \mu(x) \geq \mu(y)$ for all $x, y \in R$

Theorem 16. *A fuzzy subset μ of an ordered semihyperring R is a fuzzy ordered bi-hyperideal of R if and only if the set $\mu_t(\neq \emptyset)$ is an ordered bi-hyperideal of R for all $t \in [0, 1]$.*

Proof. The proof is similar to the proof of Theorem 12. □

Definition 17. Let $(R, +, \cdot, \leq)$ be an ordered semihyperring and let μ be a fuzzy subset of R. Then, μ is called a *fuzzy ordered quasi-hyperideal* of R if the following conditions hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y}\{\mu(z)\}$ for all $x, y \in R$

(ii) $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$

(iii) $x \leq y \Longrightarrow \mu(x) \geq \mu(y)$ for all $x, y \in R$

The following theorem can be proved in a similar way in the proof of Theorem 4.8 of [30].

Theorem 18. *A fuzzy subset μ of an ordered semihyperring R is a fuzzy ordered quasi-hyperideal of R if and only if the set $\mu_t(\neq \emptyset)$ is an ordered quasi-hyperideal of R for all $t \in [0,1]$.*

Lemma 19. *Let $(R,+,\cdot,\leq)$ be an ordered semihyperring and let χ_I be the characteristic function of I. Then, we have the following:*

(i) *I is an ordered bi-hyperideal of R if and only if χ_I is a fuzzy ordered bi-hyperideal of R*

(ii) *I is an ordered quasi-hyperideal of R if and only if χ_I is a fuzzy ordered quasi-hyperideal of R*

Proof. It is straightforward. □

Theorem 20. *Let $(R,+,\cdot,\leq)$ be an ordered semihyperring. Then, we have the following:*

(i) *Every fuzzy ordered hyperideal of R is a fuzzy ordered quasi-hyperideal of R*

(ii) *Every fuzzy ordered quasi-hyperideal of R is a fuzzy ordered bi-hyperideal of R*

Proof. (i) Only we show that the condition (ii) of Definition 17 is satisfied.

Let μ be a fuzzy ordered hyperideal of R and $a \in R$. We have

$$\left((\mu \circ 1) \cap (1 \circ \mu)\right)(a) = \min\left\{(\mu \circ 1)(a), (1 \circ \mu)(a)\right\}. \quad (17)$$

If $A_a = \emptyset$, then it is easy to see that

$$\min\left\{(\mu \circ 1)(a), (1 \circ \mu)(a)\right\} \subseteq \mu(a). \quad (18)$$

If $A_a \neq \emptyset$, then there exist $x, y \in R$ such that $a \leq x \cdot y$. Then there exists $z \in x \cdot y$ such that $a \leq z$. Since μ is a fuzzy ordered hyperideal of R, we have

$$\mu(a) \geq \mu(z) \geq \inf_{z \in x \cdot y}\{\mu(z)\} \geq \min\left\{\mu(x), \mu(y)\right\}. \quad (19)$$

That is, $\mu(a) \geq \min\{\mu(x), \mu(y)\}$. On the other hand,

$$\begin{aligned}
\left((\mu \circ 1) \cap (1 \circ \mu)\right)(a) &= \min\left\{(\mu \circ 1)(a), (1 \circ \mu)(a)\right\} \\
&= \min\left\{\sup_{(x,y)\in A_a}\left\{\min\left\{\mu(x), 1(y)\right\}\right\}, \right. \\
&\quad \left. \sup_{(x,y)\in A_a}\left\{\min\left\{1(x), \mu(y)\right\}\right\}\right\} \\
&= \min\left\{\sup_{(x,y)\in A_a}\left\{\mu(x)\right\}, \sup_{(x,y)\in A_a}\left\{\mu(y)\right\}\right\} \\
&= \sup_{(x,y)\in A_a}\left\{\min\left\{\mu(x), \mu(y)\right\}\right\} \leq \mu(a).
\end{aligned} \quad (20)$$

That is, the condition (ii) of Definition 17 is satisfied. Thus μ is a fuzzy ordered quasi-hyperideal of R.

(ii) Assume that μ is a fuzzy ordered quasi-hyperideal of R. We show that $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x \cdot y}\{\mu(z)\}$ and $\min\{\mu(x), \mu(z)\} \leq \inf_{w \in x \cdot y \cdot z}\{\mu(w)\}$ for all $x, y, z \in R$. Let $z \in x \cdot y$. We have

$$\begin{aligned}
(\mu \circ 1)(z) &= \sup_{(x,y)\in A_z} \min\left\{\mu(x), 1(y)\right\} = \mu(x) \\
(1 \circ \mu)(z) &= \sup_{(x,y)\in A_z} \min\left\{1(x), \mu(y)\right\} = \mu(y).
\end{aligned} \quad (21)$$

Since μ is a fuzzy ordered quasi-hyperideal, we have

$$\begin{aligned}
\mu(z) &\geq \left((\mu \circ 1) \cap (1 \circ \mu)\right)(z) \\
&= \min\left\{(\mu \circ 1)(z), (1 \circ \mu)(z)\right\} \\
&= \min\left\{\mu(x), \mu(y)\right\}.
\end{aligned} \quad (22)$$

Hence $z \in x \cdot y$;

$$\begin{aligned}
\inf_{z \in x \cdot y}\left\{\mu(z)\right\} &\geq \min\left\{(\mu \circ 1)(z), (1 \circ \mu)(z)\right\} \\
&\geq \min\left\{\mu(x), \mu(y)\right\}.
\end{aligned} \quad (23)$$

Similarly,

$$\begin{aligned}
\inf_{w \in x \cdot y \cdot z}\left\{\mu(w)\right\} &\geq \min\left\{(\mu \circ 1)(w), (1 \circ \mu)(w)\right\} \\
&\geq \min\left\{\mu(x), \mu(z)\right\}.
\end{aligned} \quad (24)$$

Therefore μ is a fuzzy ordered bi-hyperideal of R. □

The following example shows that the converse of Theorem 20 is not true in general.

Example 21. Consider the ordered semihyperring R which is given in Example 8. Now, it is easy to see that $A = \{0, b\}$ is an ordered bi-hyperideal of R, but it is not an ordered quasi-hyperideal of R. Let μ be a fuzzy subset of R defined by

$$\begin{aligned}
\mu(0) &= 0.7, \\
\mu(b) &= 0.7, \\
\mu(a) &= 0.4, \\
\mu(c) &= 0.4.
\end{aligned} \quad (25)$$

We have

$$\mu_t = \begin{cases} R, & \text{if } t \in (0, 0.4], \\ \{0, b\}, & \text{if } t \in (0.4, 0.7], \\ \emptyset, & \text{if } t \in (0.7, 1]. \end{cases} \quad (26)$$

Since R and $\{0, b\}$ are ordered bi-hyperideal of R, then μ_t is an ordered bi-hyperideal of R for all $t \in [0,1]$. Hence μ is a fuzzy ordered bi-hyperideal of R by Theorem 16. But it is not a fuzzy ordered quasi-hyperideal of R.

Definition 22 (see [21]). An ordered semihyperring $(R,+,\cdot,\leq)$ is called *regular*, if, for every $a \in R$, there exists $x \in R$ such that $a \leq a \cdot x \cdot a$.

Theorem 23. *Let $(R, +, \cdot, \leq)$ be a regular ordered semihyperring and let μ be a fuzzy subset of R. Then, μ is a fuzzy ordered quasi-hyperideal of R if and only if μ is a fuzzy ordered bi-hyperideal of R.*

Proof. "$\Longrightarrow$" Assume that μ is a fuzzy ordered quasi-hyperideal of R. It is clear that μ is a fuzzy ordered bi-hyperideal of R by Theorem 20 (ii).

"$\Longleftarrow$" Only we show that the condition (ii) of Definition 17 is satisfied. Let $a \in R$. If $A_a = \emptyset$, then it is easy to see that $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$. Let $A_a \neq \emptyset$.

(1) If $(\mu \circ 1)(a) \leq \mu(a)$, then we have that $\mu(a) \geq (\mu \circ 1)(a) \geq \min\{(\mu \circ 1)(a), (1 \circ \mu(a))\}$; that is, $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$.

(2) If $(\mu \circ 1)(a) > \mu(a)$, then there exists at least one pair $(z, w) \in A_a$ such that

$$\min\{\mu(z), 1(w)\} = \mu(z) > \mu(a). \tag{27}$$

That is, $z, w \in R$, $a \leq z \cdot w$, and $\mu(z) > \mu(a)$. In this case we will prove that $(1 \circ \mu)(a) \leq \mu(a)$. Let $(u, v) \in A_a$; then $a \leq u \cdot v$ for some $u, v \in R$. Since R is regular and $a \in R$ there exists $x \in R$ such that $a \leq a \cdot x \cdot a$. From $a \leq a \cdot x \cdot a$, $a \leq z \cdot w$, $a \leq u \cdot v$ we get $a \leq z \cdot w \cdot x \cdot u \cdot v$. Since μ is a fuzzy ordered bi-hyperideal R, we obtain that

$$\begin{aligned} \mu(a) &\geq \mu(z \cdot (w \cdot x \cdot u) \cdot v) \geq \inf_{\alpha \in z \cdot (w \cdot x \cdot u) \cdot v} \mu(\alpha) \\ &\geq \min\{\mu(z), \mu(v)\}. \end{aligned} \tag{28}$$

If $\min\{\mu(z), \mu(v)\} = \mu(z)$, then $\mu(a) \geq \mu(z)$. This contradicts with $\mu(z) > \mu(a)$. Hence, $\min\{\mu(z), \mu(v)\} = \mu(v)$ and so $\mu(a) \geq \mu(v) = \min\{1(u), \mu(v)\}$ for all $(u, v) \in A_a$. Therefore $\mu(a) \geq \sup_{(u,v) \in A_a} \min\{1(u), \mu(v)\} = (1 \circ \mu)(a)$. As a result,

$$\begin{aligned} ((\mu \circ 1) \cap (1 \circ \mu))(a) &= \min\{(\mu \circ 1)(a), (1 \circ \mu)(a)\} \\ &= \min\{\mu(u), \mu(v)\} = \mu(v) \leq \mu(a). \end{aligned} \tag{29}$$

That is, $(\mu \circ 1) \cap (1 \circ \mu) \subseteq \mu$. □

Now, we have to prove the main characterization theorem for regular semihyperrings. We denote by $Q(x)$ the ordered quasi-hyperideal of R generated by $x (x \in R)$.

Lemma 24. *Let R be an ordered semihyperring. Then the following conditions are equivalent:*

(i) *R is regular*

(ii) *$A = (ARA]$ for every ordered bi-hyperideal A of R*

(iii) *$Q = (QRQ]$ for every ordered quasi-hyperideal Q of R*

Proof. $(i) \Longrightarrow (ii)$ Assume that (i) holds. Let A be any ordered bi-hyperideal of R and let x be any element of A. Since R is regular, there exists $a \in R$ such that $x \leq x \cdot a \cdot x$. Then it is easy to see that $x \leq x \cdot a \cdot x \in (ARA]$. Hence $A \subseteq (ARA]$. On the other hand, since A is an ordered bi-ideal of R, we have $(ARA] \subseteq A$ and so $A = (ARA]$.$(ii) \Longrightarrow (iii)$ This proof is straightforward.

$(iii) \Longrightarrow (i)$ Assume that $Q = (QRQ]$ for every ordered quasi-hyperideal Q of R. Then $(Q(x)RQ(x)] = Q(x) \subseteq (xRx]$. Thus $x \in (xRx]$ implies that R is regular. □

Theorem 25. *Let R be an ordered semihyperring. Then, the following conditions are equivalent:*

(i) *R is regular*

(ii) *$\mu \subseteq \mu \circ 1 \circ \mu$ for every fuzzy ordered bi-hyperideal μ of R*

(iii) *$\mu \subseteq \mu \circ 1 \circ \mu$ for every fuzzy ordered quasi-hyperideal μ of R*

Proof. $(i) \Longrightarrow (ii)$ Assume that (i) holds. Let μ be any fuzzy ordered bi-hyperideal of R and $a \in R$. Since R is regular there exists $x \in R$ such that $a \leq a \cdot x \cdot a$. Hence $(ax, a) \in A_a$.

$$\begin{aligned} (\mu \circ 1 \circ \mu)(a) &= \sup_{(u,v) \in A_a} \min\{(\mu \circ 1)(u), \mu(v)\} \\ &\geq \min_{a \leq a \cdot x \cdot a} \{(\mu \circ 1)(a \cdot x), \mu(x)\} \\ &= \min\left\{\sup_{(r,s) \in A_{a \cdot x}} \{\min\{\mu(r), 1(s), \mu(a)\}\}\right\} \\ &\geq \min\{\mu(a), \mu(a)\} = \mu(a) \end{aligned} \tag{30}$$

Therefore $\mu \subseteq \mu \circ 1 \circ \mu$.

$(ii) \Longrightarrow (iii)$ This is straightforward from Theorem 20.

$(iii) \Longrightarrow (i)$ Assume that (iii) holds and let Q be any ordered quasi-hyperideal of R. Then by Lemma 19, χ_Q is a fuzzy ordered quasi-hyperideal of R. Now $\chi_Q \subseteq \chi_Q \circ 1 \circ \chi_Q = \chi_{(QRQ]}$. Therefore $Q \subseteq (QRQ]$. On the other hand, since Q is any ordered quasi-hyperideal of R, we have $(QRQ] \subseteq Q$ and so $Q = (QRQ]$. By Lemma 24, R is regular. □

5. Conclusion

In the structural theory of fuzzy algebraic systems, fuzzy ideals with special properties always play an important role. In this paper, we study fuzzy ordered hyperideals, fuzzy ordered quasi-hyperideals, and fuzzy ordered bi-hyperideals of an ordered semihyperring. We characterize regular ordered semihyperrings by the properties of these fuzzy hyperideals. As a further work, we will also concentrate on characterizations of different classes of ordered semihyperrings in terms of fuzzy interior hyperideals.

Data Availability

All data generated or analysed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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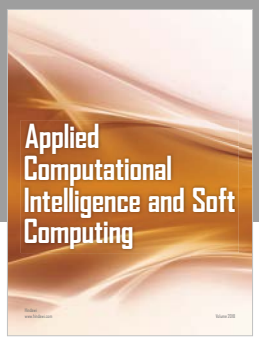
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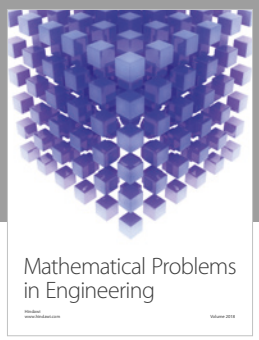
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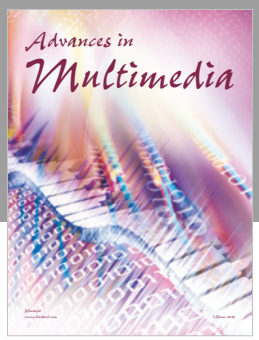
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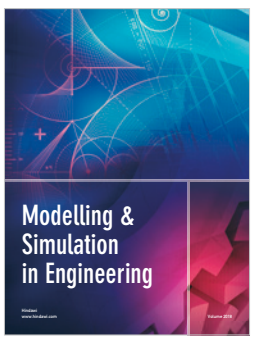


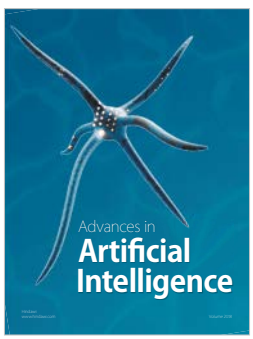




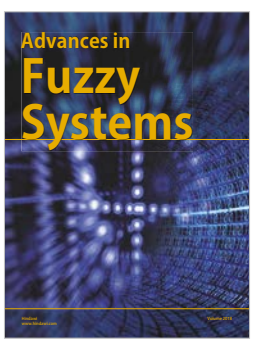








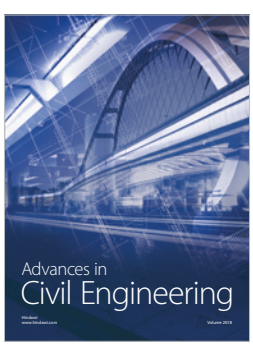


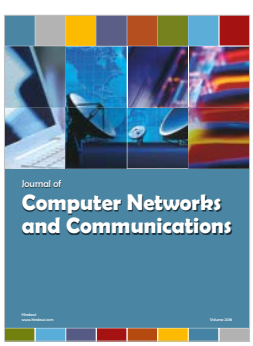


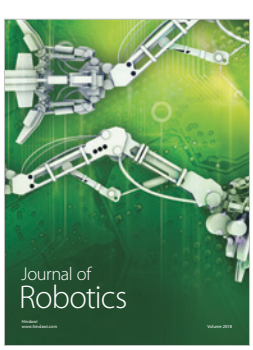








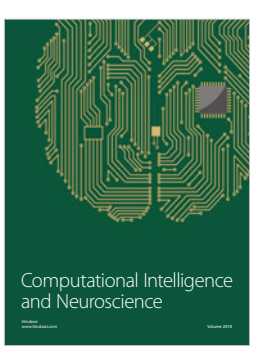












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