ON EIGENVALUES AND EIGENVECTORS IN $M_n(\mathbb{R})$

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ABSTRACT

This study is about main structure of eigenvalues and eigenvectors. Eigenvalues and eigenvectors have many applications. Armin Schwartzman, Walter F. Mascarenhas and Jonathan E. Taylor made an evaluation in the affine subspace in 2014 in the study titled "Inference for eigenvalues and eigenvectors of Gaussian symmetric matrices". The basis of this calculation is based on the solution of systems of equations. This solution yields parallel vectors in the plane. The $\square^3$ space also exhibits the same situation. The fixed square matrix can correspond to different eigenvalues and different eigenvectors. This is a known situation. The case of non-parallel vectors obtained in the plane or in space is discussed. Examples are given and new methods are explored. Common matrix and common eigenvalue cases are considered. The resulting results are carefully studied. Researchers Andrew J. Liounis , John A. Christian , and Shane B. Robinson contributed to the subject with the study "Observations on the Computation of Eigenvalue and Eigenvector Jacobians". These authors contributed by taking the aforementioned subject in a functional hand. In my study, it is observed how the common elements, which will form the basis of the subject, exhibited. Solutions are made in the plane. Graphs with examples are given. New approaches is introduced in this regard. The groundwork is prepared for the next phases of the subject. Some cases resulting from multiplication of matrices and affecting the solution are given. That is, the contribution of division operation in matrices in this case is noted. New results, examples and features is transferred. The basis for the subject of the next study is prepared.

Keywords: eigenvalues, eigenvectors, matrices, equation.

1. INTRODUCTION

Eigenvalues and eigenvectors are used to solve many problems such as engineering and health. Vibration problems in solving finite analysis probes and determination of the optimum parameter vector are given as examples. There are many articles that have been studied on this subject, perhaps because of the practical solutions this suject provides to the application areas of this subject. Studies on eigenvalues and eigenvectors are grouped into three categories [1]. This study is more about a new approach to calculating eigenvalues and eigenvectors.

Let us start with the eigenvalue problem defined in the literature.

The set of all matrices of order n over a field F is denoted by $M_n(F)$.

$A=\left[a_{ij}\right]_n \in M_n(F), \vec{u} \in V_n(F)$ and $\lambda \in F$

$$A\vec{u}=\lambda\vec{u}$$

The eigenvalue and eigenvector calculations are obtained from the solution of the given equation below. Let the matrix A be given. If there are the eigenvalues of this matrix, then these eigenvalues are found by known algebraic methods. An eigenvector $\vec{u}$ corresponds to each eigenvalue λ [2].

In this study, the procedure is performed for $n=2$.

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \left|A-\lambda I_2\right|=0$$

$$\lambda^2-\lambda\left(a_{11}+a_{22}\right)+a_{11}a_{22}-a_{12}a_{21}=0$$

$$\lambda_{1,2}=\frac{1}{2}a_{11}+a_{22}\pm\frac{1}{2}\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}}$$

$$\left(A-\lambda I_2\right)\vec{u}=\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\lambda_{1,2}=\frac{1}{2}a_{11}+a_{22}+\frac{1}{2}\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}}$, then

$$\begin{bmatrix} a_{11}-\frac{1}{2}a_{11}+a_{22}+\frac{1}{2}\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}} & a_{12} \\ a_{21} & a_{22}-\frac{1}{2}a_{11}+a_{22}+\frac{1}{2}\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}} \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{2}u_{12}\left(a_{11}+2a_{12}u_{12}-a_{22}-\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}}\right)=0$$

$$-\frac{1}{2}u_{12}\left(a_{11}-2a_{12}-a_{22}+\sqrt{{a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}}\right)=0$$

$${a_{11}}^2-2a_{11}a_{22}+{a_{22}}^2+4a_{12}a_{21}=\left(a_{11}-a_{22}\right)^2+4a_{12}a_{21}\geq 0$$

$$|a_{11}-a_{22}| \geq -4a_{12}a_{21}, \text{ where } a_{12}a_{21} \in \mathbb{R}.$$

2. BETWEEN THE COMPONENTS OF THE MATRIX AND THE COMPONENTS OF THE EIGENVECTORS

Some cases between the components of eigenvalues and eigenvectors with known methods are examined [4].

Property 2.1. Let $A \in M_2(\mathbb{R})$.

A. Between the components of the matrix and the components of the eigenvectors, the followings hold for $\lambda_1 = \frac{1}{2}a_{11} + a_{22} + \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}$.

i.
$$u_{11} \in \begin{cases} \left\{ \dfrac{2a_{12}u_{12}}{-a_{11}+a_{22}+\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}} \right\}, If\ \frac{1}{2}a_{11} - \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} \neq 0 \\ \mathbb{C}, If\ -a_{12}u_{12} = 0, \frac{1}{2}a_{11} - \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} = 0 \\ \varnothing, If\ -a_{12}u_{12} \neq 0, \frac{1}{2}a_{11} - \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} = 0 \end{cases}$$

ii.
$$u_{11} \in \begin{cases} \left\{ \dfrac{u_{12}}{2a_{21}}\left(a_{11} - a_{22} + \sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) \right\}, If\ a_{21} \neq 0 \\ \mathbb{C}, If\ a_{21} = 0, u_{12}\left(-\frac{1}{2}a_{11} + \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) = 0 \\ \varnothing, If\ a_{21} = 0, u_{12}\left(-\frac{1}{2}a_{11} + \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) \neq 0 \end{cases}$$

B. Between the components of the matrix and the components of the eigenvectors, the followings hold for $\lambda_2 = \frac{1}{2}a_{11} + a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}$.

i.
$$u_{11} \in \begin{cases} \left\{ -\dfrac{2a_{12}u_{12}}{a_{11}-a_{22}+\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}} \right\}, If\ \frac{1}{2}a_{11} - \frac{1}{2}a_{22} + \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} \neq 0 \\ \mathbb{C}, If\ -a_{12}u_{12} = 0, \frac{1}{2}a_{11} - \frac{1}{2}a_{22} + \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} = 0 \\ \varnothing, If\ -a_{12}u_{12} \neq 0, \frac{1}{2}a_{11} - \frac{1}{2}a_{22} + \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}} = 0 \end{cases}$$

ii.
$$u_{11} \in \begin{cases} \left\{ -\dfrac{u_{12}}{2a_{21}}\left(-a_{11} + a_{22} + \sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) \right\}, If\ a_{21} \neq 0 \\ \mathbb{C}, If\ a_{21} = 0, u_{12}\left(\frac{1}{2}a_{11} - \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) = 0 \\ \varnothing, If\ a_{21} = 0, u_{12}\left(\frac{1}{2}a_{11} - \frac{1}{2}a_{22} - \frac{1}{2}\sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}\right) \neq 0 \end{cases}$$

Lemma 2.1. Let $A \in M_2(\mathbb{R})$. Then

$$u_{11} = -\frac{a_{12}}{a_{21}} u_{12}.$$

Proof. Let $A \in M_2(\mathbb{R})$. By Property 2.1,

If

$$u_{11} = -\frac{2a_{21}u_{21}}{a_{11} - a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}},$$

$$u_{21} = \frac{2a_{21}u_{11}}{a_{11} - a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}},$$

Then

$$u_{11} = -\frac{a_{12}}{a_{21}} u_{12}.$$

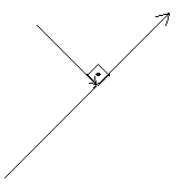
Example 1.3. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ then $|A - \lambda I_2| = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$$\lambda_1 = 1 \rightarrow \vec{u}_1 = x \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x \in \mathbb{R}\backslash\{0\},$$

$$\lambda_3 = 3 \rightarrow \vec{u}_1 = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x \in \mathbb{R}\backslash\{0\}.$$

3. ON EIGENVALUES AND EIGENVECTORS IN $M_n(\mathbb{R})$

The calculation principle of eigenvectors is based on the orthogonality of the row vectors of the matrix $A - \lambda I_n = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}_n$ and the column vectors of the matrix $U = \left[u_{ij}\right]_n$ [5,6].



By example 1.3 for $\lambda_1 = 1$

$$A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in M_2(\mathbb{R}), U = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in M_2(\mathbb{R}).$$

And for $\lambda_2 = 3$

$$A - \lambda I_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in M_2(\mathbb{R}), U = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \in M_2(\mathbb{R}).$$

4. RESULT

Let $A=\left[a_{ij}\right]_n$ be a matrix.

i. If the corresponding eigenvalues λ vector is $\vec{u}$ then
$$\left(A-\lambda I_n\right)\left[\vec{u}\ \ t_1\vec{u}\ \ \cdots\ \ t_n\vec{u}\right]_{n\times r}=\left[0\right]_{n\times r}.$$

ii. If $X=\left[x_{ij}\right]_{n\times r}$ then the eigenvalues are same and
$$AX=\lambda X\Rightarrow AX-\lambda X=\left(A-\lambda I_n\right)X=\left[0\right].$$

iii. If $Y=\left[y_{ij}\right]_{r\times n}$ then the eigenvalues are same and
$$YA=\lambda Y\Rightarrow YA-\lambda Y=Y\left(A-\lambda I_n\right)=\left[0\right]$$ (See [3]).

5. DISCUSSIONS

In the study of the components of the eigenvalues and eigenvectors, a part of the study was not reflected due to the process bloat.

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