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Construction methods for interval-valued t-norms on bounded lattices

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Abstract
In this paper, some construction methods for interval-valued t-norms (t-conorms) based on a down-set or an upset are presented. It is worth noting that the new construction methods do not have to be t-representable, which is a significant difference between the construction methods proposed in this paper and the commonly used construction methods of interval-valued t-norms (t-conorms) on other truth values sets. Some illustrative examples and remarks are also included to clarify the methods.

Keywords Interval-valued t-norms · Interval-valued t-conorms · Down-sets · Up-sets · Bounded lattices

Mathematics Subject Classification 03B52 · 03E72

1 Introduction

The history of triangular norms (t-norms in short) dates back to the 1940s. These special aggregation operations were introduced by Menger (1942) in which they were used when generalizing the triangle inequality. The axiomatic definition of t-norms, as we use today, is given by Schweizer and Sklar (1958): a t-norm is an increasing, commutative and associative binary function on [0, 1] with the neutral element 1. Then, instead of the unit interval [0, 1], t-norms have been studied on some more general structures, such as bounded lattices, posets,

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interval-valued sets (see (Klement et al. 2000; Karaçal and Khadjiev 2005; Ertuğrul et al. 2019; Karaçal et al. 2019; Ertuğrul et al. 2015; Aşici 2022; Karaçal and Şanlı 2022; Bedregal et al. 2023; Cao and Liu 2026)).

Interval-valued sets are a generalization of fuzzy sets in which each element of the universe is assigned to a closed subinterval of the unit interval that approximately determines its unknown degree of membership. Since it is very difficult to determine the exact value of the membership degree in most real life problems, it is much more reasonable to work on interval-valued sets. For this reason, it is very important to study generalizations of classical conjunctions such as t-norms, t-conorms, etc., on interval-valued sets. Moreover, since interval-valued set theory has applications in many different areas such as soft computing, approximate reasoning, optimization (see (Karaçal et al. 2024; Luo et al. 2024, 2020; Gorzałczany 1987; Bustince 2010; Gayen et al. 2026; Câmara and Jafelice 2026; Bustince et al. 2025)), the construction methods of interval-valued fuzzy logic operators has attracted the interest of researchers.

In Deschrijver (2008), it has recently been shown that $\mathcal{T}_{T_1,T_2}$ for any t-norms T_1, T_2 on a bounded lattice L such that $T_1 \leq T_2$ is an interval-valued t-norm on the bounded lattice $L^I = \{[x_1, x_2] \mid x_1, x_2 \in L \text{ and } x_1 \leq x_2\}$, and it is called the t-representable t-norm.

$$\mathcal{T}_{T_1,T_2}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)] \text{ for all } [x_1, x_2], [y_1, y_2] \in L^I \quad (1)$$

Therefore, a t-representable t-norm on the L^I can be generated by the formula (1) via t-norms T_1, T_2 on L satisfying $T_1 \leq T_2$. In Deschrijver (2008), it was also shown that t-norms on L^I satisfying the condition $T(x, y \vee z) = T(x, y) \vee T(x, z)$ for all $x, y, z \in L^I$ and two border conditions can be represented by a t-norm on L and a unary function on L. In addition, some examples of non-t-representable t-norms are given in the same study. In order to obtain a characterization of t-norms on L^I, it is necessary to obtain a characterization of non-t-representable t-norms on L^I. For this reason, it is very important to investigate the methods of obtaining non-representable t-norms on L^I.

We aim with this paper to introduce construction methods for interval-valued t-norms (t-conorms) and show the fact that the interval-valued t-norms (t-conorms) obtained from the mentioned methods do not have to be t-representable (t-co-representable). First, we propose two construction methods for interval-valued t-norms, via an up-set or a down-set, and two t-norms with some idempotent elements on the bounded lattice L. We also exemplify the methods with exploratory examples. Besides, it is shown that our construction methods do not have to be t-representable. Moreover, we dually introduce interval-valued t-conorm construction methods by means of a down-set or an up-set, and t-conorms with idempotent elements on a bounded lattice L. We show that these construction methods do not have to be t-co-representable and support the theory with examples. In addition to providing illustrative examples, we elaborate the construction methods in some special cases with some corollaries.

This paper is organized as follows. Section 2 is devoted to some main definitions and results, which we will use for our paper. In the next section, we recall the interval-valued t-norm and t-conorm construction methods in the literature. In Section 4, we present new construction methods for interval-valued t-norms and t-conorms. Moreover, these methods are supported with the explanatory examples. Finally, we end the paper with Section 5, concluding remarks.

2 Preliminaries

First, we recall main definitions which are required for the rest of the paper.

Let L be a set and $\leq$ be a binary relation on L. $\leq$ is called a partial order relation on L if it satisfies reflexivity, antisymmetry and transitivity properties. Also, L is called a partially ordered set (poset) denoted by $(L, \leq)$ if the binary relation $\leq$ is a partial order relation on L.

Let $(L, \leq)$ be a poset and $A \subseteq L$. An element $a \in L$ is called an upper bound of A if $x \leq a$ for all $x \in A$. Similarly, the lower bound of A can be defined.

The supremum (infumum) of A, if it exists, is the least upper (greatest lower) bound of A and denoted by $\bigvee A$ ($\bigwedge A$). Especially, if $A = \{a, b\}$, the supremum of A is denoted by $a \vee b$ and the infumum of A is denoted by $a \wedge b$.

Let L be a non-empty set and $(L, \leq)$ be a poset. L is called a lattice if for all pair of elements in L the supremum and infimum exist, that is $x \vee y,\ x \wedge y \in L$ for arbitrary $x, y \in L$.

Let $(L, \leq)$ be a lattice. The elements x and y are called comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable and the notation $x \parallel y$ is used for such elements. L is called a bounded lattice, denoted by $(L, \leq, 0_L, 1_L)$, if it has the smallest element and greatest element, denoted by 0_L and 1_L, respectively.

Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A subset P of L is called a down-set if $x \in P$, $y \in L$ and $y \leq x$ imply $y \in P$. Similarly a subset F of L is called an up-set if $x \in F$, $y \in L$ and $x \leq y$ imply $y \in F$.

We refer to Birkhoff (1967) for more information.

2.1 Basic informations for t-norms and t-conorms on L

In this section we recall some definitions, properties and examples for t-norms and t-conorms on a bounded lattice L.

Definition 1 Klement et al. (2000) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A t-norm is a function $T : L^2 \longrightarrow L$ satisfying, for all $x, y, z \in L$,

T1) $T(x, y) = T(y, x)$ (commutativity).
T2) $T(x, 1) = x$ (neutral element).
T3) If $y \leq z$, then $T(x, y) \leq T(x, z)$ (monotonicity).
T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

Definition 2 Klement et al. (2000) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A t-conorm is a function $S : L^2 \longrightarrow L$ satisfying, for all $x, y, z \in L$,

S1) $S(x, y) = S(y, x)$ (commutavity).
S2) $S(x, 0_L) = x$ (neutral element).
S3) If $y \leq z$, then $S(x, y) \leq S(x, z)$ (monotonicity).
S4) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity).

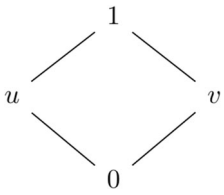
Example 1 Klement et al. (2000) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. Two basic t-norms T_D and $T_\wedge$ on a bounded lattice L are respectively defined for all $x, y \in L$ as follows:

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1_L, \\ x & \text{if } y = 1_L, \\ 0_L & \text{otherwise,} \end{cases}$$

and

$$T_\wedge(x, y) = x \wedge y.$$

Fig. 1 The lattice L



Two basic t-conorms S_D and $S_\vee$ on a bounded lattice L are respectively defined for all $x, y \in L$ as follows:

$$S_D(x, y) = \begin{cases} y & \text{if } x = 0_L, \\ x & \text{if } y = 0_L, \\ 1_L & \text{otherwise,} \end{cases}$$

and

$$S_\vee(x, y) = x \vee y.$$

Let T_1 (S_1) and T_2 (S_2) be two t-norms (t-conorms) on L. T_1 (S_1) is called smaller than T_2 (S_2) and is denoted by $T_1 \leq T_2$ ($S_1 \leq S_2$) if for any $x, y \in L$, $T_1(x, y) \leq T_2(x, y)$ ($S_1(x, y) \leq S_2(x, y)$). In this sense, the smallest and greatest t-norms (t-conorms) on a bounded lattice L are given respectively by T_D ($S_\vee$) and $T_\wedge$ (S_D).

Definition 3 Çaylı et al. (2016) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $a \in L$ and $T : L^2 \longrightarrow L$ ($S : L^2 \longrightarrow L$) be a t-norm (a t-conorm). a is called idempotent element of T (S) if $T(a, a) = a$ ($S(a, a) = a$).

2.2 T-norms (t-conorms) on L^I

In this section, we recall some definitions, properties, and examples for t-norms and t-conorms on the bounded lattice L^I.

Definition 4 Deschrijver (2008) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. We define $(L^I, \leq_{L^I})$ by

$$L^I = \{[x_1, x_2] \mid x_1, x_2 \in L \text{ and } x_1 \leq x_2\},$$

$$[x_1, x_2] \leqslant_{L^I} [y_1, y_2] \Longleftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2 \quad \text{for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.$$

One can observe that L^I is a lattice, and the smallest and the greatest elements of L^I are $[0_L, 0_L]$ and $[1_L, 1_L]$, denoted by $\mathbf{0}_{L^I}$ and $\mathbf{1}_{L^I}$ respectively, where 0_L and 1_L are the external elements of L. Therefore, $(L^I, \leq_{L^I}, \mathbf{0}_{L^I}, \mathbf{1}_{L^I})$ is a bounded lattice.

In the following example, we apply Definition 4 for a bounded lattice.

Example 2 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1. We have the lattice L^I with order given in Fig. 2 by applying Definition 4.

The smallest and greatest t-norms are, respectively, defined on L^I for any $[x_1, x_2], [y_1, y_2] \in L^I$ as follows:

$$\mathcal{T}_D([x_1, x_2], [y_1, y_2]) = \begin{cases} [y_1, y_2] & \text{if } [x_1, x_2] = [1_L, 1_L], \\ [x_1, x_2] & \text{if } [y_1, y_2] = [1_L, 1_L], \\ [0_L, 0_L] & \text{otherwise,} \end{cases}$$

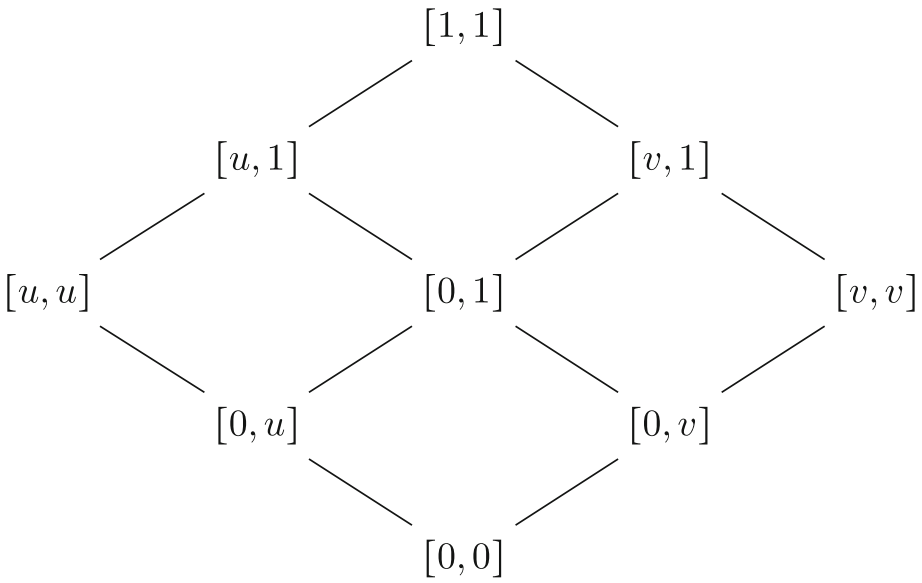


Fig. 2 The lattice L^I

and

$$\mathcal{T}_{\wedge}([x_1, x_2], [y_1, y_2]) = [x_1 \wedge x_2, y_1 \wedge y_2].$$

In the literature, there are some methods to obtain a t-norm on L^I. One of the well-known methods is presented by the following theorem.

Theorem 1 *Deschrijver (2008) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $T_1, T_2 : L^2 \longrightarrow L$ be two t-norms on L. Then, the function $\mathcal{T}_{T_1,T_2} : (L^I)^2 \to L^I$ defined by, for all $[x_1, x_2], [y_1, y_2] \in L^I$,*

$$\mathcal{T}_{T_1,T_2}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)], \tag{2}$$

is a t-norm on L^I if and only if $T_1 \leq T_2$.

The t-norm $\mathcal{T}_{T_1,T_2}$ given by formula (2) is called t-representable t-norm on L^I. Clearly, if a t-norm $\mathcal{T}$ on L^I is a t-representable, then $\mathcal{T}([0_L, 1_L], [x_1, x_2]) = [0_L, x_2]$ for any $[x_1, x_2] \in L^I$.

In Deschrijver (2008), some other construction methods are also included to produce t-norms that are not t-representable on L^I. We list that methods with Example 3, as follows.

Example 3 Deschrijver (2008) Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. Note that not all t-norms on L^I are t-representable.

1. Let T be a t-norm on L. Then the mapping $\mathcal{T}_T : \left(L^I\right)^2 \to L^I$ defined by, for all $[x_1, x_2], [y_1, y_2] \in L^I$,

$$\mathcal{T}_T([x_1, x_2], [y_1, y_2]) = \left[T(x_1, y_1), T(x_1, y_2) \vee T(x_2, y_1)\right],$$

is a t-norm on L^I, called a pseudo-t-representable t-norm. For any pseudo-t-representable t-norm $\mathcal{T}_T$ we have that $\mathcal{T}_T\left([0_L, 1_L], [0_L, 1_L]\right) = [0_L, 0_L]$, which shows that $\mathcal{T}_T$ is not t-representable.

2. Let T be a t-norm on L and $t \in L$. Then the mapping $\mathcal{T}_{T,t} : \left(L^I\right)^2 \to L^I$ defined by, for all $[x_1, x_2], [y_1, y_2] \in L^I$,

$$\mathcal{T}_{T,t}([x_1, x_2], [y_1, y_2]) = \Big[T(x_1, y_1), T\big(t, T(x_2, y_2)\big) \vee T(x_1, y_2) \vee T(x_2, y_1)\Big],$$

is a non-t-representable t-norm on L^I if $t \neq 1_L$ and also it is called a generalized pseudo-t-representable t-norm. Note that $\mathcal{T}_{T,t}([0_L, 1_L], [0_L, 1_L]) = [0_L, t]$. If $t = 0_L$, then $\mathcal{T}_{T,t} = \mathcal{T}_T$ is a pseudo-t-representable t-norm; if $t = 1_L$, $\mathcal{T}_{T,t} = \mathcal{T}_{T,T}$.

The greatest and smallest t-conorms are, respectively, defined on L^I for all $[x_1, x_2], [y_1, y_2] \in L^I$ as follows:

$$\mathcal{S}_D([x_1, x_2], [y_1, y_2]) = \begin{cases} [y_1, y_2] & \text{if } [x_1, x_2] = [0_L, 0_L], \\ [x_1, x_2] & \text{if } [y_1, y_2] = [0_L, 0_L], \\ [1_L, 1_L] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{S}_\vee([x_1, x_2], [y_1, y_2]) = [x_1 \vee y_1, x_2 \vee y_2].$$

Theorem 2 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $S_1, S_2 : L^2 \longrightarrow L$ be two t-conorms on L. Then, the function $\mathcal{S}_{S_1,S_2} : (L^I)^2 \to L^I$ defined by, for all $[x_1, x_2], [y_1, y_2] \in L^I$,*

$$\mathcal{S}_{S_1,S_2}([x_1, x_2], [y_1, y_2]) = [S_1(x_1, y_1), S_2(x_2, y_2)], \tag{3}$$

is a t-conorm on L^I if and only if $S_1 \leq S_2$.

The t-conorm $\mathcal{S}_{S_1,S_2}$ is called as t-co-representable t-conorm on L^I. Clearly, if a t-conorm $\mathcal{S}$ on L^I is a t-co-representable, then $\mathcal{S}([0_L, 1_L], [x_1, x_2]) = [x_1, 1_L]$ for any $[x_1, x_2] \in L^I$.

3 Some construction methods for t-norms on L^I by the down-set or the up-set of L

In this section, we introduce some construction methods for interval-valued t-norms by exploiting an up-set or a down-set. Moreover, we exemplify the theories with some examples, and also we elaborate these methods with corollaries. Considering the examples and results, it is obtained that the interval-valued t-norms given by formulas (4) and (8) do not need to be t-representable.

In the following theorem, we present a construction method for t-norms on L^I. In this theorem, we exploit a down-set of the bounded lattice L, a t-norm T_1 with the idempotent element a and a t-norm T_2 with the idempotent element b such that $a \leq b$ and $T_1 \leq T_2$. Also, formula (4) in Theorem 3 produces family of interval-valued t-norms dependent on the elements a and b of L. In this respect, the construction methods differ from the construction method given by the formula (2) in the Theorem 1.

Theorem 3 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, P be a down-set with $\varnothing \neq P \subseteq L\backslash\{1_L\}$, $T_1 : L^2 \longrightarrow L$ be a t-norm with an idempotent element a and $T_2 : L^2 \longrightarrow L$ be a t-norm with an idempotent element b such that $a \leq b$ and $T_1 \leq T_2$. Then, the function $\mathcal{T}^k_{a,b} : (L^I)^2 \longrightarrow L^I$*

defined by, for $k \in \{1, 2\}$ *and for all* $[x_1, x_2], [y_1, y_2] \in L^I$,

$$\mathcal{T}^k_{a,b}([x_1, x_2], [y_1, y_2]) = \begin{cases} [y_1, y_2] & \text{if } [x_1, x_2] = [1_L, 1_L], \\ [x_1, x_2] & \text{if } [y_1, y_2] = [1_L, 1_L], \\ [T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))] & \text{if } [x_1, x_2], [y_1, y_2] \neq [1_L, 1_L] \\ & \text{and } T_k(x_k, y_k) \in P, \\ [T_1(x_1, y_1), T_2(x_2, y_2)] & \text{otherwise}, \end{cases} \tag{4}$$

is a t-norm on L^I.

Proof Let's prove for $k = 1$. Similarly, the proof can be done for $k = 2$ in a similar fashion. Observe that the commutativity of $\mathcal{T}^1_{a,b}$ and the fact that $[1_L, 1_L]$ is a neutral element of $\mathcal{T}^1_{a,b}$ are obvious from the definition of $\mathcal{T}^1_{a,b}$.

Monotonicity ($\mathcal{T}_3$): Suppose $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^I$ and $[y_1, y_2] \leq_{L^I} [z_1, z_2]$. We need to show that $\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) \leq_{L^I} \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2])$. If $[x_1, x_2] = [1_L, 1_L]$ or $[y_1, y_2] = [1_L, 1_L]$, the proof is immediate. All other possible cases are listed below.

1. Let $[x_1, x_2] \neq [1_L, 1_L]$, $[y_1, y_2] \neq [1_L, 1_L]$ and $[z_1, z_2] = [1_L, 1_L]$.

1.1. If $T_1(x_1, y_1) \in P$, then

$$\begin{aligned} \mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) &= [T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))] \\ &\leq_{L^I} [T_1(x_1, y_1), T_2(x_2, y_2)] \\ &\leq_{L^I} [x_1, x_2] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2]). \end{aligned}$$

1.2. If $T_1(x_1, y_1) \notin P$, then

$$\begin{aligned} \mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) &= [T_1(x_1, y_1), T_2(x_2, y_2)] \\ &\leq_{L^I} [x_1, x_2] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2]). \end{aligned}$$

2. Let $[x_1, x_2], [y_1, y_2], [z_1, z_2] \neq [1_L, 1_L]$ and $T_1(x_1, y_1) \in P$.

2.1. If $T_1(x_1, z_1) \in P$, then

$$\begin{aligned} \mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) &= [T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))] \\ &\leq_{L^I} [T_1(a, T_1(x_1, z_1)), T_2(b, T_2(x_2, z_2))] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2]). \end{aligned}$$

2.2. If $T_1(x_1, z_1) \notin P$, then

$$\begin{aligned}\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) &= [T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))] \\ &\leq_{L^I} [T_1(x_1, y_1), T_2(x_2, y_2)] \\ &\leq_{L^I} [T_1(x_1, z_1), T_2(x_2, z_2)] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2]).\end{aligned}$$

3. Let $[x_1, x_2], [y_1, y_2], [z_1, z_2] \neq [1_L, 1_L]$ and $T_1(x_1, y_1) \notin P$. Since $T_1(x_1, y_1) \leq T_1(x_1, z_1)$ and $T_1(x_1, y_1) \notin P$, it has to $T_1(x_1, z_1) \notin P$. Under these situations,

$$\begin{aligned}\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]) &= [T_1(x_1, y_1), T_2(x_2, y_2)] \\ &\leq_{L^I} [T_1(x_1, z_1), T_2(x_2, z_2)] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [z_1, z_2]).\end{aligned}$$

Associativity ($\mathcal{T}_4$): We should show that $\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) = \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2]))$ for all $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^I$ to prove the associativity. If $[1_L, 1_L] \in \{[x_1, x_2], [y_1, y_2], [z_1, z_2]\}$, the proof is trivial. Thus, we suppose $[1_L, 1_L] \notin \{[x_1, x_2], [y_1, y_2], [z_1, z_2]\}$. The proof is split into all possible cases.

1. Let $T_1(x_1, y_1) \in P$.

1.1. If $T_1(y_1, z_1) \in P$, then

$$\begin{aligned}&\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) \\ &= \mathcal{T}^1_{a,b}([T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))], [z_1, z_2]) \\ &= [T_1(a, T_1(T_1(a, T_1(x_1, y_1)), z_1)), T_2(b, T_2(T_2(b, T_2(x_2, y_2)), z_2))] \\ &= [T_1(a, T_1(T_1(x_1, y_1)), z_1)), T_2(b, T_2(T_2(x_2, y_2)), z_2))] \\ &= [T_1(a, T_1(x_1, T_1(y_1, z_1))), T_2(b, T_2(x_2, T_2(y_2, z_2)))] \\ &= [T_1(a, T_1(x_1, T_1(a, T_1(y_1, z_1)))), T_2(b, T_2(x_2, T_2(b, T_2(y_2, z_2))))] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [T_1(a, T_1(y_1, z_1)), T_2(b, T_2(y_2, z_2))]) \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2])).\end{aligned}$$

1.2. If $T_1(y_1, z_1) \notin P$, then

$$\begin{aligned}&\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) \\ &= \mathcal{T}^1_{a,b}([T_1(a, T_1(x_1, y_1)), T_2(b, T_2(x_2, y_2))], [z_1, z_2]) \\ &= [T_1(a, T_1(T_1(a, T_1(x_1, y_1)), z_1)), T_2(b, T_2(T_2(b, T_2(x_2, y_2)), z_2))] \\ &= [T_1(T_1(a, T_1(x_1, y_1)), z_1), T_2(T_2(b, T_2(x_2, y_2)), z_2))] \\ &= [T_1(a, T_1(x_1, T_1(y_1, z_1))), T_2(b, T_2(x_2, T_2(y_2, z_2)))] \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], [T_1(y_1, z_1), T_2(y_2, z_2)]) \\ &= \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2])).\end{aligned}$$

2. Let $T_1(x_1, y_1) \notin P$.
2.1. Let $T_1(T_1(x_1, y_1), z_1) \in P$.
2.1.1. If $T_1(y_1, z_1) \in P$, then

$$\begin{aligned}
&\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) \\
&= \mathcal{T}^1_{a,b}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2]) \\
&= [T_1(a, T_1(T_1(x_1, y_1), z_1)), T_2(b, T_2(T_2(x_2, y_2), z_2))] \\
&= [T_1(a, T_1(x_1, T_1(y_1, z_1))), T_2(b, T_2(x_2, T_2(y_2, z_2))] \\
&= [T_1(T_1(a, a), T_1(x_1, T_1(y_1, z_1))), T_2(T_2(b, b), T_2(x_2, T_2(y_2, z_2))] \\
&= [T_1(a, T_1(x_1, T_1(a, T_1(y_1, z_1)))), T_2(b, T_2(x_2, T_2(b, T_2(y_2, z_2))))] \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], [T_1(a, T_1(y_1, z_1)), T_2(b, T_2(y_2, z_2))]) \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2])).
\end{aligned}$$

2.1.2. If $T_1(y_1, z_1) \notin P$, then

$$\begin{aligned}
&\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) \\
&= \mathcal{T}^1_{a,b}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2]) \\
&= [T_1(a, T_1(T_1(x_1, y_1), z_1)), T_2(b, T_2(T_2(x_2, y_2), z_2))] \\
&= [T_1(a, T_1(x_1, T_1(y_1, z_1))), T_2(b, T_2(x_2, T_2(y_2, z_2)))] \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], [T_1(y_1, z_1), T_2(y_2, z_2)]). \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2])).
\end{aligned}$$

2.2. Let $T_1(T_1(x_1, y_1), z_1) \notin P$.

$$\begin{aligned}
&\mathcal{T}^1_{a,b}(\mathcal{T}^1_{a,b}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) \\
&= \mathcal{T}^1_{a,b}([T_1(x_1, y_1), T_2(x_2, y_2)], [z_1, z_2]) \\
&= [T_1(T_1(x_1, y_1), z_1), T_2(T_2(x_2, y_2), z_2)] \\
&= [T_1(x_1, T_1(y_1, z_1)), T_2(T_2(x_2, T_2(y_2, z_2))] \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], [T_1(y_1, z_1), T_2(y_2, z_2)]) \\
&= \mathcal{T}^1_{a,b}([x_1, x_2], \mathcal{T}^1_{a,b}([y_1, y_2], [z_1, z_2])).
\end{aligned}$$

Summarizing all the above profs, $\mathcal{T}^1_{a,b}$ is a t-norm on L^I. □

Remark 1 Consider the t-norm $\mathcal{T}^k_{a,b}$ on L^I given by Theorem 3 with $k = 1$. Whenever $b \neq 1$, $\mathcal{T}^1_{a,b}$ is a t-norm on L^I which is not t-representable.

Example 4 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1. Let the t-norms T_1, T_2 be t-norm T given in Table 1 the down-set $P = \{0, v\}$. By applying the formula (4) in Theorem 3 with $k = 2$, $a = b = u$, we obtain the t-norm $\mathcal{T}^2_{u,u}$ on L^I as in Table 2.

Note that, we have $\mathcal{T}^2_{u,u}([0, 1], [v, v]) = [0, 0] \neq [0, v]$ from the Table 2, which shows that the t-norm $\mathcal{T}^2_{a,b}$ given by Theorem 3 does not have to be t-representable.

Table 1 The t-norm T on L

T	0	u	v	1
0	0	0	0	0
u	0	u	0	u
v	0	0	0	v
1	0	u	v	1

Table 2 The t-norm $\mathcal{T}^2_{u,u}$ on L^I

$\mathcal{T}^2_{u,u}$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$[0,u]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,u]$	$[0,u]$	$[0,u]$	$[0,u]$
$[0,v]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,v]$
$[u,u]$	$[0,0]$	$[0,u]$	$[0,0]$	$[u,u]$	$[0,0]$	$[0,u]$	$[u,u]$	$[0,u]$	$[u,u]$
$[v,v]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[v,v]$
$[0,1]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[u,1]$	$[0,0]$	$[0,u]$	$[0,0]$	$[u,u]$	$[0,0]$	$[0,1]$	$[u,1]$	$[0,1]$	$[u,1]$
$[v,1]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,u]$	$[0,0]$	$[0,1]$	$[0,1]$	$[0,1]$	$[v,1]$
$[1,1]$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$

We elaborate on Theorem 3 in the following Corollary 1, making chocies for the elements a, b and the t-norm T. Especially, it is worth nothing that $\mathcal{T}^k_{1,1}$ is the greatest t-norm on L^I when $T_1, T_2 = T_\wedge$.

Corollary 1 *Let $T_1, T_2 = T_\wedge$ in Theorem 3.*

1. *The t-norm $\mathcal{T}^k_{a,b}$ given by the formula (4) is obtained as follows:*

$$\mathcal{T}^k_{a,b}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [1_L,1_L], \\ [a \wedge x_1 \wedge y_1, b \wedge x_2 \wedge y_2] & \text{if } [x_1,x_2],[y_1,y_2] \neq [1_L,1_L] \\ & \text{and } x_k \wedge y_k \in P, \\ [x_1 \wedge y_1, x_2 \wedge y_2] & \text{otherwise.} \end{cases} \tag{5}$$

2. *If $a = b = 1_L$, then it is obtained the greatest t-norm on L^I.*
3. *If $a = b = 0_L$, then the t-norm $\mathcal{T}^k_{a,b}$ given by the formula (4) is obtained as follows:*

$$\mathcal{T}^k_{0_L,0_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [1_L,1_L], \\ [0_L,0_L] & \text{if } [x_1,x_2],[y_1,y_2] \neq [1_L,1_L] \text{ and } x_k \wedge y_k \in P, \\ [x_1 \wedge y_1, x_2 \wedge y_2] & \text{otherwise.} \end{cases} \tag{6}$$

4. *If $a = 0_L$ and $b = 1_L$, then the t-norm $\mathcal{T}^k_{a,b}$ given by the formula (4) is obtained as follows:*

$$\mathcal{T}^k_{0_L,1_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[1_L,1_L], \\ [0_L,x_2\wedge y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[1_L,1_L] \text{and } x_k\wedge y_k\in P, \\ [x_1\wedge y_1,x_2\wedge y_2] & \text{otherwise.} \end{cases} \tag{7}$$

After Corollary 1, it is quite normal for one to wonder whether the smallest t-norm on L can be obtained by $a = b = 0_L$ and $T_1, T_2 = T_D$. We answer this question negatively with the following Remark 2.

Remark 2 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1, the t-norms $T_1, T_2 = T_D$, and the down-set $P = \{0, v\}$. By applying the formula (4) in Theorem 3 with $k = 2$, $a = b = 0$, we obtain the t-norm $\mathcal{T}^2_{0,0}([0,1],[0,1]) = [0,1]$ but we know $\mathcal{T}_{\mathcal{D}}([0,1],[0,1]) = [0,0]$. Thus $\mathcal{T}^2_{0,0} \neq \mathcal{T}_{\mathcal{D}}$.

In the following theorem, we give another construction method for t-norms on L^I.

Theorem 4 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, F be an up-set with $\varnothing \neq F \subseteq L\backslash\{0_L\}$, $T_1 : L^2 \longrightarrow L$ be a t-norm with an idempotent element a and $T_2 : L^2 \longrightarrow L$ be a t-norm with an idempotent element b such that $a \leq b$ and $T_1 \leq T_2$. Then, the function $\mathcal{T}^{k,*}_{a,b} : (L^I)^2 \longrightarrow L^I$ defined by, for $k \in 1, 2$ and for all $[x_1,x_2],[y_1,y_2] \in L^I$,*

$$\mathcal{T}^{k,*}_{a,b}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[1_L,1_L], \\ [T_1(x_1,y_1),T_2(x_2,y_2)] & \text{if } [x_1,x_2],[y_1,y_2]\neq[1_L,1_L] \\ & \text{and } T_k(x_k,y_k)\in F, \\ [T_1(a,T_1(x_1,y_1)),T_2(b,T_2(x_2,y_2))] & \text{otherwise} \end{cases} \tag{8}$$

is a t-norm on L^I.

Proof The proof follows by Theorem 3 from the fact that $L\backslash F$ is a down set. □

Remark 3 Consider the t-norm $\mathcal{T}^{k,*}_{a,b}$ on L^I given by Theorem 4 with $k = 1$. Whenever $b \neq 1_L$, $\mathcal{T}^{1,*}_{a,b}$ is a t-norm on L^I which is not t-representable.

Example 5 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1, the t-norms $T_1, T_2 = T_\wedge$, and the up-set $F = \{v, 1\}$. By applying the formula (8) in Theorem 4 $a = 0$, we obtain the t-norm $\mathcal{T}^{2,*}_{0,v}$ on L^I as in Table 3.

By means of this example, we have $\mathcal{T}^{2,*}_{0,v}([0,1],[u,u]) = [0,0] \neq [0,u]$ from the Table 3, which shows that the t-norm $\mathcal{T}^{2,*}_{a,b}$ given by Theorem 4 does not have to be t-representable.

Choosing the idempotent elements $a, b \in L$ and the corresponding t-norms T_1, T_2, respectively, a detailed discussion of Theorem 4 is given in the following corollary. Also, note that the formula (8) produces the greatest t-norm on L^I when $a = b = 1_L$ and $T_1, T_2 = T_\wedge$.

Table 3 The t-norm $\mathcal{T}_{0,v}^{2,*}$ on L^I

$\mathcal{T}_{0,v}^{2,*}$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$[0,u]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,u]$
$[0,v]$	$[0,0]$	$[0,0]$	$[0,v]$	$[0,0]$	$[0,v]$	$[0,v]$	$[0,v]$	$[0,v]$	$[0,v]$
$[u,u]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[u,u]$
$[v,v]$	$[0,0]$	$[0,0]$	$[0,v]$	$[0,0]$	$[v,v]$	$[0,v]$	$[0,v]$	$[v,v]$	$[v,v]$
$[0,1]$	$[0,0]$	$[0,0]$	$[0,v]$	$[0,0]$	$[0,v]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
$[u,1]$	$[0,0]$	$[0,0]$	$[0,v]$	$[0,0]$	$[0,v]$	$[0,1]$	$[u,1]$	$[0,1]$	$[u,1]$
$[v,1]$	$[0,0]$	$[0,0]$	$[0,v]$	$[0,0]$	$[v,v]$	$[0,1]$	$[0,1]$	$[v,1]$	$[v,1]$
$[1,1]$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$

Corollary 2 *Let $T_1, T_2 = T_\wedge$ in Theorem 4.*

1. *The t-norm $\mathcal{T}_{a,b}^{k,*}$ given by the formula (8) is rewritten as follows:*

$$\mathcal{T}_{a,b}^{k,*}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[1_L,1_L], \\ [x_1\wedge y_1, x_2\wedge y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[1_L,1_L] \\ & \text{and } x_k\wedge y_k\in F, \\ [a\wedge x_1\wedge y_1, b\wedge x_2\wedge y_2] & \text{otherwise.} \end{cases} \tag{9}$$

2. *If $a=b=0_L$, then the t-norm $\mathcal{T}_{a,b}^{k,*}$ given by the formula (8) is rewritten as follows:*

$$\mathcal{T}_{0,0}^{k,*}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[1_L,1_L], \\ [x_1\wedge y_1, x_2\wedge y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[1_L,1_L] \text{ and } x_k\wedge y_k\in F, \\ [0_L,0_L] & \text{otherwise.} \end{cases} \tag{10}$$

3. *If $a=b=1_L$, it is obtained the greatest t-norm on L^I.*
4. *If $a=0_L$ and $b=1_L$, then the t-norm $\mathcal{T}_{a,b}^{k,*}$ given by the formula (8) is rewritten as follows:*

$$\mathcal{T}_{0,1}^{k,*}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[1_L,1_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[1_L,1_L], \\ [x_1\wedge y_1, x_2\wedge y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[1_L,1_L] \text{ and } x_k\wedge y_k\in F, \\ [0_L, x_2\wedge y_2] & \text{otherwise.} \end{cases} \tag{11}$$

In the following Remark 4 shows that the formula (8) does not the smallest t-norm on L^I even if $a=b=0_L$ and $T_1, T_2 = T_D$.

Table 4 The t-conorm S on L

S	0	u	v	1
0	0	u	v	1
u	u	u	1	1
v	v	1	1	1
1	1	1	1	1

Remark 4 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1, the t-norms $T_1, T_2 = T_D$, and the up-set $F = \{v, 1\}$ .By applying the formula (8) in Theorem 4 with $k = 2$, $a = b = 0$, we have $\mathcal{T}^{2,*}_{0,0}([0, 1], [0, 1]) = [0, 1]$ but we know $\mathcal{T}_{\mathcal{D}}([0, 1], [0, 1]) = [0, 0]$. Thus $\mathcal{T}^{2,*}_{0,0} \neq \mathcal{T}_{\mathcal{D}}$.

4 Some construction methods for t-conorms on L^I by the down-set or the up-set of L

In this section, we introduce some construction methods for interval-valued t-conorms, without proofs, which are based on an up-set or a down-set. We show that the t-conorms obtained by the construction methods do not have to be t-co-representable.

In the following theorem, we give a method via an up-set, a t-conorm S_1 with an idempotent element a and a t-conorm S_2 with an idempotent element b such that $a \leq b$ and $S_1 \leq S_2$ to construct interval-valued t-conorms.

Theorem 5 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, F be an up-set with $\varnothing \neq F \subseteq L\backslash\{0_L\}$, $S_1 : L^2 \longrightarrow L$ be a t-conorm with an idempotent element a and $S_2 : L^2 \longrightarrow L$ be a t-conorm with an idempotent element b such that $a \leq b$ and $S_1 \leq S_2$. Then, the function $\mathcal{S}^k_{a,b} : (L^I)^2 \longrightarrow L^I$ defined by, for $k \in \{1, 2\}$ and for all $[x_1, x_2], [y_1, y_2] \in L^I$,*

$$\mathcal{S}^k_{a,b}([x_1, x_2], [y_1, y_2]) = \begin{cases} [y_1, y_2] & \text{if } [x_1, x_2] = [0_L, 0_L], \\ [x_1, x_2] & \text{if } [y_1, y_2] = [0_L, 0_L], \\ [S_1(a, S_1(x_1, y_1)), S_2(b, S_2(x_2, y_2))] & \text{if } [x_1, x_2], [y_1, y_2] \neq [0_L, 0_L] \\ & \text{and } S_k(x_k, y_k) \in F, \\ [S_1(x_1, y_1), S_2(x_2, y_2)] & \text{otherwise.} \end{cases} \tag{12}$$

is a t-conorm on L^I.

Remark 5 Consider the t-conorm $\mathcal{S}^k_{a,b}$ on L^I given by Theorem 5 with $k = 2$. Whenever $a \neq 0_L$, $\mathcal{S}^2_{a,b}$ is a t-conorm on L^I which is not t-co-representable.

Now, let us illustrate the following example as an application of Theorem 5.

Example 6 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1. Let the t-conorms S_1, S_2 be S given in Table 4 and the up-set $F = \{v, 1\}$. By applying the formula (12) in Theorem 5 with $k = 1, a = u, b = 1$, we obtain the t-conorm $\mathcal{S}^1_{u,1}$ on L^I as in Table 5.

Table 5 The t-conorm $\mathcal{S}^1_{u,1}$ on L^I

$\mathcal{S}^1_{u,1}$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,u]$	$[0,u]$	$[0,u]$	$[0,1]$	$[u,u]$	$[1,1]$	$[0,1]$	$[u,1]$	$[1,1]$	$[1,1]$
$[0,v]$	$[0,v]$	$[0,1]$	$[0,1]$	$[u,1]$	$[1,1]$	$[0,1]$	$[u,1]$	$[1,1]$	$[1,1]$
$[u,u]$	$[u,u]$	$[u,u]$	$[u,1]$	$[u,u]$	$[1,1]$	$[u,1]$	$[u,1]$	$[1,1]$	$[1,1]$
$[v,v]$	$[v,v]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[u,1]$	$[1,1]$	$[0,1]$	$[u,1]$	$[1,1]$	$[1,1]$
$[u,1]$	$[u,1]$	$[u,1]$	$[u,1]$	$[u,1]$	$[1,1]$	$[u,1]$	$[u,1]$	$[1,1]$	$[1,1]$
$[v,1]$	$[v,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$

It is obtained $\mathcal{S}^1_{u,1}([0,1],[v,v]) = [1,1] \neq [v,1]$, which shows that the t-conorm $\mathcal{S}^1_{a,b}$ given by Theorem 5 does not have to be t-co-representable.

In the following corollary, we examine in Theorem 5 in detail when $S_1, S_2 = S_\vee$.

Corollary 3 *Let $S_1, S_2 = S_\vee$ in Theorem 5.*

1. *The t-conorm $\mathcal{S}^k_{a,b}$ given by the formula (12) is obtained as follows:*

$$\mathcal{S}^k_{a,b}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [0_L,0_L], \\ [a \vee x_1 \vee y_1, b \vee x_2 \vee y_2] & \text{if } [x_1,x_2],[y_1,y_2] \neq [0_L,0_L] \\ & \text{and } x_k \vee y_k \in F, \\ [x_1 \vee y_1, x_2 \vee y_2] & \text{otherwise.} \end{cases} \tag{13}$$

2. *If $a = b = 1_L$, the t-conorm $\mathcal{S}^k_{a,b}$ given by the formula (12) is obtained as follows:*

$$\mathcal{S}^k_{1_L,1_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [0_L,0_L], \\ [1_L,1_L] & \text{if } [x_1,x_2],[y_1,y_2] \neq [0_L,0_L] \text{and } x_k \vee y_k \in F, \\ [x_1 \vee y_1, x_2 \vee y_2] & \text{otherwise.} \end{cases} \tag{14}$$

3. *If take $a = b = 0_L$, then it is obtained the smallest t-conorm on L^I.*
4. *If take $a = 0_L$ and $b = 1_L$, the t-conorm $\mathcal{S}^k_{a,b}$ given by the formula (12) is obtained as follows:*

$$\mathcal{S}^k_{0_L,1_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [0_L,0_L], \\ [x_1 \vee y_1, 1_L] & \text{if } [x_1,x_2],[y_1,y_2] \neq [0_L,0_L] \text{ and } x_k \vee y_k \in F, \\ [x_1 \vee y_1, x_2 \vee y_2] & \text{otherwise.} \end{cases} \tag{15}$$

Table 6 The t-conorm $\mathcal{S}^{1,*}_{v,v}$ on L^I

$\mathcal{S}^{1,*}_{v,v}$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,u]$	$[0,v]$	$[u,u]$	$[v,v]$	$[0,1]$	$[u,1]$	$[v,1]$	$[1,1]$
$[0,u]$	$[0,u]$	$[0,u]$	$[0,1]$	$[1,1]$	$[v,1]$	$[0,1]$	$[1,1]$	$[v,1]$	$[1,1]$
$[0,v]$	$[0,v]$	$[0,1]$	$[0,v]$	$[1,1]$	$[v,v]$	$[0,1]$	$[1,1]$	$[v,1]$	$[1,1]$
$[u,u]$	$[u,u]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[v,v]$	$[v,v]$	$[v,1]$	$[v,v]$	$[1,1]$	$[v,v]$	$[v,1]$	$[1,1]$	$[v,1]$	$[1,1]$
$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[1,1]$	$[v,1]$	$[0,1]$	$[1,1]$	$[v,1]$	$[1,1]$
$[u,1]$	$[u,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[v,1]$	$[v,1]$	$[v,1]$	$[v,1]$	$[1,1]$	$[v,1]$	$[v,1]$	$[1,1]$	$[v,1]$	$[1,1]$
$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$

We obtain the smallest t-conorm on L^I from the formula (12) choosing $a = b = 0_L$ and $S_1, S_2 = S_\vee$. However, the formula (12) does not need to produce the greatest t-conorm on L^I. In the following Remark 6 reveals that the formula (12) does not produce the greatest t-conorm on L^I even if $a = b = 1_L$ and the t-conorms $S_1, S_2 = S_D$.

Remark 6 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1, the t-conorms $S_1, S_2 = S_D$, and the up-set $F = \{v, 1\}$. By applying the formula (8) in Theorem 4 when $k = 1$, $a = b = 1$, we have $\mathcal{S}^1_{1,1}([0,1],[0,1]) = [0,1]$ but we know $\mathcal{S}_{\mathcal{D}}([0,1],[0,1]) = [1,1]$. Thus $\mathcal{S}^1_{1,1} \neq \mathcal{S}_{\mathcal{D}}$.

In the following theorem, we present another construction method.

Theorem 6 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, P be a down-set $\varnothing \neq P \subseteq L\backslash\{1_L\}$, $S_1 : L^2 \longrightarrow L$ be a t-conorm with an idempotent element a and $S_2 : L^2 \longrightarrow L$ be a t-conorm with an idempotent element b such that $a \leq b$ and $S_1 \leq S_2$. Then, the function $\mathcal{S}^{k,*}_{a,b} : (L^I)^2 \longrightarrow L^I$ defined by, for $k \in \{1,2\}$ and for all $[x_1,x_2],[y_1,y_2] \in L^I$,*

$$\mathcal{S}^{k,*}_{a,b}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2] = [0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2] = [0_L,0_L], \\ [S_1(x_1,y_1), S_2(x_2,y_2)] & \text{if } [x_1,x_2],[y_1,y_2] \neq [0_L,0_L] \\ & \text{and } S_k(x_k,y_k) \in P, \\ [S_1(a,S_1(x_1,y_1)), S_2(b,S_2(x_2,y_2))] & \text{otherwise,} \end{cases} \tag{16}$$

is a t-conorm on L^I.

Example 7 Consider the bounded lattice $L = \{0, u, v, 1\}$ described in Fig. 1. Let the t-conorms S_1, S_2 be $S_\vee$ and the down-set $P = \{0, v\}$. By applying the formula (16) in Theorem 6 with $k = 1$, $a = b = v$, we obtain the t-conorm $\mathcal{S}^{1,*}_{v,v}$ on L^I as in Table 6.

Remark 7 Consider the t-conorm $\mathcal{S}^{k,*}_{a,b}$ on L^I given by Theorem 6 and $k = 2$. Whenever $a \neq 0_L$, $\mathcal{S}^{2,*}_{a,b}$ is a t-conorm on L^I which is not t-co-representable.

Similar to the Corollary 3, we obtain the following results.

Corollary 4 *Let $S_1, S_2 = S_\vee$ in Theorem 6.*

1. *The t-conorm $\mathcal{S}^{k,*}_{a,b}$ given by the formula (16) is rewritten as follows:*

$$\mathcal{S}^{k,*}_{a,b}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[0_L,0_L], \\ [x_1\vee y_1, x_2\vee y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[0_L,0_L] \\ & \text{and } x_k\vee y_k\in P, \\ [a\vee x_1\vee y_1, b\vee x_2\vee y_2] & \text{otherwise.} \end{cases} \tag{17}$$

2. *If $a=b=1_L$, then the t-conorm $\mathcal{S}^{k,*}_{a,b}$ given by the formula (16) is rewritten as follows:*

$$\mathcal{S}^{k,*}_{1_L,1_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[0_L,0_L], \\ [x_1\vee y_1, x_2\vee y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[0_L,0_L] \text{ and } x_k\vee y_k\in P, \\ [1_L,1_L] & \text{otherwise.} \end{cases} \tag{18}$$

3. *If take $a=b=0_L$, then it is obtained the smallest t-conorm on L^I.*
4. *If $a=0_L$ and $b=1_L$, then the t-conorm $\mathcal{S}^{k,*}_{a,b}$ given by the formula (16) is rewritten as follows:*

$$\mathcal{S}^{k,*}_{0_L,1_L}([x_1,x_2],[y_1,y_2]) = \begin{cases} [y_1,y_2] & \text{if } [x_1,x_2]=[0_L,0_L], \\ [x_1,x_2] & \text{if } [y_1,y_2]=[0_L,0_L], \\ [x_1\vee y_1, x_2\vee y_2] & \text{if } [x_1,x_2],[y_1,y_2]\neq[0_L,0_L] \text{ and } x_k\vee y_k\in P, \\ [x_1\vee y_1, 1_L] & \text{otherwise.} \end{cases} \tag{19}$$

We conclude that it can be obtained the smallest t-conorm on L^I by means of formula (16) when $a=b=0_L$ and the t-conorms $S_1, S_2 = S_\vee$. However, even if $a=b=1_L$ and the t-conorms $S_1, S_2 = S_D$ is chosen, the greatest t-conorm may not be produced by via formula (16). In order show that, we give the following Remark 8

Remark 8 Consider the bounded lattice $L=\{0,u,v,1\}$ described in Fig. 1. By applying the formula (16) in Theorem 6 with $k=1$, $a=b=1$, the down-set $P=\{0,v\}$ and the t-conorms $S_1, S_2 = S_D$, we have $\mathcal{S}^{1,*}_{1,1}([0,1],[0,1]) = [0,1] \neq [1,1] = \mathcal{S}_D([0,1],[0,1])$ and thus $\mathcal{S}^{1,*}_{1,1} \neq \mathcal{S}_D$.

5 Concluding remarks

In this paper, we discussed construction methods of interval-valued t-norms (t-conorms) which not supposed to be t-representable (t-co-representable). First, we proposed two construction methods for interval-valued t-norms by using an up-set or a down-set of L, and t-norms T_1, T_2 having idempotent element a, b, respectively. More specifically, we added some examples for a better understanding of these construction methods. Furthermore, we

revealed in which cases the construction methods are t-representable and in which cases they are not. Moreover, we deeply investigated our new construction methods for t-norms on L^I with the corresponding corollaries. Dually, we presented construction methods for interval-valued t-conorms and we explained the methods with examples and corollaries.

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Data availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflicts of Interest The authors declare no conflict of interest.

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