ON MAIN PROPERTIES OF THE POLOID AND THE GROUP

Keleş HASAN [1]

[1] Karadeniz Technical University, Faculty of Science, ORCID ID: 0000-0001-8436-890X

ABSTRACT

In this study, some features between a grup and a poloid are examined. The study is concerned with the simple group structure and the poloid structure. The group is defined as the mathematical structure that provides the properties of closure, the associativity, the invertibility, and the identy by an operation defined on a set. This mathematical structure is the plainest and simplest form of mathematics. Many other mathematical studies are based on this construction. Group Theory has many applications. Some of those; The solution of Rubik's cube with Group Theory, modeling of crystals and hydrogen atoms using symmetry groups, and the use of this in many fields of Physics, Chemistry and Engineering sciences.

The Poloid is obtained by adding the fifth property, escort elements, to the group structure. This property is made additional contributions to the existing situation. The non-commutative group is brought to a new structure with this feature element escort. Examples of matrices that implement the poloid property are given.

Keywords: Group, poloid, escort element, poloid feature.

1. INTRODUCTION

The theory of algebraic equations, number theory and geometry are among the main reasons for the emergence of group theory. The basis of the definition of poloid is based on the solution of matrix linear equations [1, 3].

Poloid Theory will offer a wider perspective to the application areas of Group Theory.

The group definition and the poloid definition differ only in one feature (P4). Therefore, the (P1), (P2) and (P3) properties and $(P,*)$ structure correspond to the group definition in [1, 4, 6, 7].

A group is a set P equipped with a binary operation $*: P\times P\to P$ that associates an element $a*b\in P$ to every pair of elements $a,b\in P$, and having the following properties: $*$ is associative, has an identity element $e\in P$, and every element in P is invertible More explicitly, this means that the following equations hold for all $a,b,c,d\in P$:

$P1.\ a*(b*c)=(a*b)*c$, (associativity).

$P2.\ a*e=e*a$, (identity).

$P3.$ For every $a \in P$, there is some $a^{-1} \in P$ such that $a * a^{-1} = a^{-1} * a = e$ (inverse).
$P4.$ For every $a \in P \setminus \{e\}$, there are some $b, c, d \in P \setminus \{e\}$ such that $b * c = c * d = a$ with $b \neq d$ (escort).

A set P together with an operation $*: P \times P \to P$ and an element e satisfying only conditions (P1), (P2), (P3) and (P4) is called *a poloid*. It is denoted by $(P, *)$ in [2, 3,].
Some elements are commutative in a poloid, while all elements are commutative in a commutative group.

The following Lemma 1.1. is given without proof.
Lemma 1.1. (See [1, 2, 3, 4, 6]). Let $(P, *)$ be a poloid and for all $a, b, c, d \in P$ the followings hold.
(i) If e' is a second such unit element, then

$$e' = e.$$

(ii) If $b * a = e$ and $a * c = e$, then

$$b = c.$$

(iii) If $a * b = a * c$ and $b * a = c * a$, then

$$b = c.$$

(iv) If $e \notin \{a, b, c\}$, $b * a = c$ and $a * d = c$, then

$$b \neq d.$$

An example of the structure wich is most suitable for the definition of poloid is regular matrices. The set of these regular matrices form a poloid structure according to the multiplication operation.The set of all matrices of order n over a field F is denoted by $M_n(F)$.

Theorem 1.1. (See [2, 6, 7]). Let $A, B, X \in M_n(F)$ be such that A, B and X regular matrices and X an unknown matrix. Then, in the solution of the equation $AX = B$, there are regular matrices $A = B_2 A_3$, $B = B_2 B_3$, such as $B_2, B_3, A_3 \in M_n(F)$, and the rational matrix $\frac{B_3}{A_3}$ is the solution of the equation $AX = B$. This solution is equal to the rational matrix $\frac{B}{A}$ in [3].

Matrices obtained as a result of these operations are also regular.

2. GOUP AND POLOID

Because any poloid is built on a group, there is a group within every poloid. The (P4) condition is required for a group to form a poloid.

Theorem 1.1 extends the matrices to poloids.

Theorem 2.1. Let $(P,*)$ be a poloid. For all $a \in P$ and $i, j, k = 1, ..., n; n \in Z^+$, $\exists p_i, p_j, p_k \in P$, then

$$a = p_i * p_j,\ a = p_j * p_k, \text{ where } p_i \neq p_j \neq p_k \text{ for } i \neq j \neq k .$$

Proof. In a poloid, each element is written in terms of another element.

Corollary 2.2. As a result of the above theorem, Let $a_i, a_k \in P$. Then,

(i) $e = a_i * a_k$.

(ii) $a_i^{-1} = a_k$.

Lemma 2.3 Let $(P,*)$ be a poloid. If $a = a^{-1} * p_i, a^{-1} = a * q_i$ is $\exists p_i, q_i \in P$, for $\forall a \in P \setminus \{e\}$. Then,

$$p_i * q_i = q_i * p_i = e .$$

Proof. Let $(P,*)$ be a poloid. If $a = a^{-1} * p_i, a^{-1} = a * q_i$ for $\forall a \in P \setminus \{e\}$. Then,

$$a * a^{-1} = (a^{-1} * p_i) * (a * q_i) = e$$

$$(a^{-1} * p_i)^{-1} = a * q_i$$

$$p_i^{-1} * a = a * q_i \Leftrightarrow p_i^{-1} = q_i$$

$$p_i * q_i = q_i * p_i = e .$$

Örnek 2.4. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \in M_3(\ )$. Then $A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = A$$

$1\ 4\ 2\ 4\ 3$

P_i

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{6} \\ -\frac{1}{18} & \frac{2}{9} & \frac{1}{6} \\ \frac{1}{18} & -\frac{2}{9} & \frac{1}{3} \end{bmatrix}}_{Q_i} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} = A^{-1}$$

$$P_iQ_i = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{6} \\ -\frac{1}{18} & \frac{2}{9} & \frac{1}{6} \\ \frac{1}{18} & -\frac{2}{9} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ in [5].}$$

Some definitions and lemmas given for the group are written for the poloid.

Definition 2.5.([2, 3]). Let $n \in \mathbb{Z}^+$, let let $(P,*)$ be a poloid and let $a \in P$. Then we define a^n as follows:

$$a^n = \begin{cases} \underbrace{a*a*\cdots*a}_{n}, If\ n>0 \\ \underbrace{a^{-1}*a^{-1}*\cdots*a^{-1}}_{n}, If\ n<0 \\ e, If\ n=0 \end{cases}$$

Proposition 2.6.([2, 3]) . Let $m,n \in \mathbb{Z}^+$, and $(P,*)$ be a poloid. Then

i. $a^m * a^n = a^{m+n}$.

ii. $(a^m)^n = a^{mn}$.

Lemma 2.7. Let $(P,*)$ be a poloid. For any $a \in P \setminus \{e\}$, there exist $p_1, q_{1i} \in P \setminus \{e\}$ such that $a = p_1^{n-1} * q_{1(n-1)}$, where $n \in \mathbb{Z}^+, i = 1,\ldots,n$.

Proof. Let $(P,*)$ be a poloid. For any $a \in P \setminus \{e\}$, there are $p_1, q_{1i} \in P \setminus \{e\}$:

$$a = p_1 * q_{1i}.$$
$$q_{1i} = p_1 * q_{12}$$
$$a = p_1^2 * q_{12}$$
$$\vdots$$
$$a = p_i^n * q_{1n}.$$

Örnek 2.8. For $A=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}\in M_2(¡)$. Let $P_1=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}, Q_{11}=\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}\in M_2(¡)$. Then

$$P_1Q_{11}=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}=A$$

$$Q_{11}=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}-5 & 17\\ 4 & -13\end{bmatrix}=\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}$$

$$\underbrace{1\ 4\ 2\ 4\ 3}_{Q12}$$

$$P_1^2Q_{12}=\left(\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\right)^2\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}=A$$

$$\underbrace{1\ 4\ 2\ 4\ 3}_{Q12}$$

$$\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}17 & -56\\ -13 & 43\end{bmatrix}=Q_{12}$$

$$\underbrace{1\ 4\ 2\ 4\ 3}_{Q12}\qquad\underbrace{1\ 4\ 2\ 4\ 3}_{Q_{13}}$$

$$P_1^3Q_{13}=\left(\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\right)^3\begin{bmatrix}17 & -56\\ -13 & 43\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}=A$$

$$\underbrace{1\ 4\ 2\ 4\ 3}_{Q_{13}}$$

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$$P_1^nQ_{1n}=\left(\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\right)^3[\text{K}]=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}=A.$$

$$\underbrace{\}}_{Q_{13}}$$

Some results of the above Lemma 2.7 are given below.

Corollary 2.9. Let the conditions for Lemma 2.7 be satisfied. Then the followings hold.

i. There is only one element $q_{1j}\in P$ such that $p_1q_{1j}=e$, where $1<j<n$.

ii. $p_1^{-1}=p_1^{\,j}$, where $1<j<n$.

3. RESULTS AND DISCUSSION

The (P4) condition, which gives the group the feature of being poloid, creates a relationship between the elements of a group that are different from the unit and the other elements. This creates new properties between the elements and the operation. The poloid structure is open to research in terms of many features as well as some operational features related to group structure.

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